# Diffusion scaling of a limit-order book model

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# Outline

1. "Zero-intelligence" Poisson models of the limit-order book.
2. Partial history.
3. Our model.
4. The diffusion limit of our model.
   - Split Brownian motion
   - Snapped Brownian motion
5. Areas for future research.

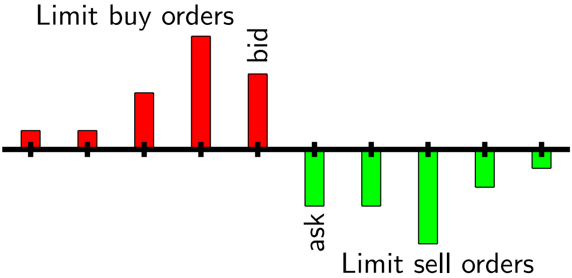
# Limit-order book

Orders to buy and sell an asset arrive at an exchange.

1. Market buy/sell order — specifies number of shares to be bought/sold at the best available price, right away.
2. Limit buy/sell order — specifies a price and a number of shares to be bought/sold at that price, when possible.
3. Order cancellation — agents who have submitted a limit order may cancel the order before it is executed.

- Market orders are executed immediately.
- Limit orders are queued for later execution, but may cancel.
- The Limit-Order Book is the collection of queued limit orders awaiting execution or cancellation.

# Limit-order book: Bid and ask prices




- The bid price is the highest limit buy order price in the book. It is the best available price for a market sell.
- The ask price is the lowest limit sell order price in the book. It is the best available price for a market buy.

# Goal of this talk

Build a "zero-intelligence" Poisson model of the limit-order book and determine its diffusion limit.

- "Zero-intelligence" — No strategic play by the agents submitting orders.
- Poisson — Arrivals of buy and sell limit and market orders are Poisson processes. Exponentially distributed waiting times before cancellations.
- Maybe a little intelligence — Arrival and cancellation rates depend on the state of the limit-order book.
- Diffusion scaling — Accelerate time by a factor of $n$, divide volume by $\sqrt{n}$, and pass to the limit as $n \to \infty$.
- Diffusion limit — Evolution of the limiting limit-order book is described in terms of Brownian motions.

# Partial history:

- GARMAN, M. (1976) "Market microstructure," *J. Financial Economics* **3**, 257–275

Poisson arrivals of buy and sell orders, rather than a model based on utility maximization. Arrival rates depend on price, which is set by a market maker to maximize his profit.

- SMITH, E., FARMER, J.D., GILLEMOT, L. & KRISHNAMURTHY, S. (2003) "Statistical theory of the continuous double auction," *Quant. Finance* **3**, 481–514.

Poisson arrivals of buy and sell orders and exponential waiting times before cancellations. Simulation studies. Get realistic predictions about price volatility, market depth, bid-ask spread, price impact of placing a market order, etc.

# Partial history:

- Bayraktar, E., Horst, U. & Sircar, R. (2008) "Queueing theoretic approaches to financial price fluctuations," in *Handbooks in OR & MS*, Vol. 15, J. R. Birge and V. Linetsky, eds., pp. 637–677.

A survey of "zero-intelligence modeling," including work by the same authors in which long-range dependence of prices is obtained as a diffusion-scaled limit of a semi-Markov model with inert investors.

- Cont, R., Stoikov, S. & Talreja, R. (2010) "A stochastic model for order book dynamics," *Operations Research* **58**, 549–563.

Poisson arrivals of buy and sell orders keyed off the opposite best price. Use Laplace transform analysis to compute statistics, e.g., probability of an increase in price, probability of making the spread.

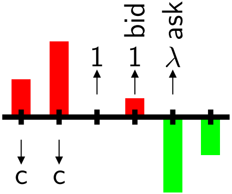
# Partial history:

- Cont, R. & de Larrard, A. (2013) "Price dynamics in a Markovian limit order market," *SIAM J. Financial Mathematics* **4**, 1–25.

Bid and ask always differ by one tick, and orders queue only at the bid and ask. If one of these is depleted, both move one tick and the book reinitializes. Derive the diffusion-scaled limit.

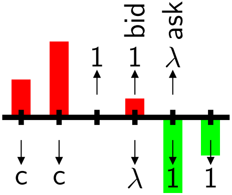
- This talk
  - We derive the diffusion-scaled limit.
  - The order book consists of more than queues at bid and ask.
  - We continue through the price change without reinitializing the model.
  - Our limiting model has a two-tick spread at almost every time, contrary to empirical observations.
  - We do not yet have a mathematically rigorous proof for every claim.

# Our model: Arrivals and cancellations of buy orders



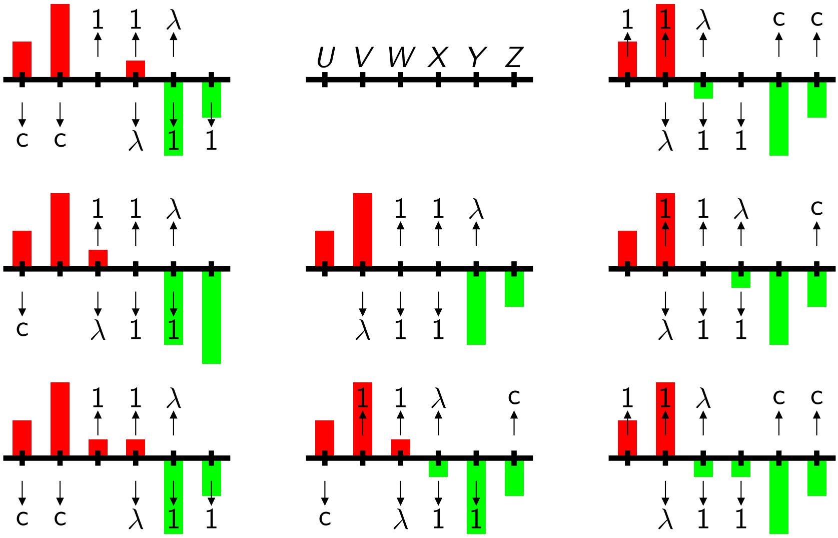

- All arriving and departing orders are of size 1.
- Poisson arrivals of market buys at rate $\lambda > 1$. These execute at the ask price.
- Poisson arrivals of limit buys at one and two ticks below the ask price, both at rate 1.
- Cancellations of limit buys two or more ticks below the bid price, at rate $\theta/\sqrt{n}$ per order.
- All processes are independent of one another.

# Our model: Arrivals and cancellations of sell orders



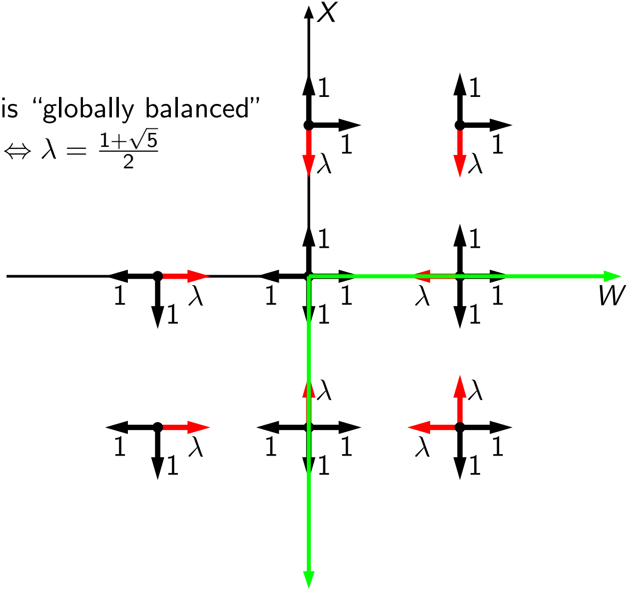

- All orders are of size 1.
- Poisson arrivals of market sells at rate $\lambda > 1$. These execute at the bid price.
- Poisson arrivals of limit sells at one and two ticks above the bid price, both at rate 1.
- Cancellations of limit sells two or more ticks above the ask price, at rate $\theta/\sqrt{n}$ per order.
- All processes are independent of one another.

# Our model: Limit-order book arrivals and departures



# Split Brownian motion: Transitions of $(W, X)$

$(W, X)$ is "globally balanced"
$\Leftrightarrow \lambda = \frac{1+\sqrt{5}}{2}$



# Split Brownian motion: Theorem

The diffusion scaling of a generic process $Q$ is defined to be

$$\widehat{Q}_n(t) := \frac{1}{\sqrt{n}} Q(nt).$$

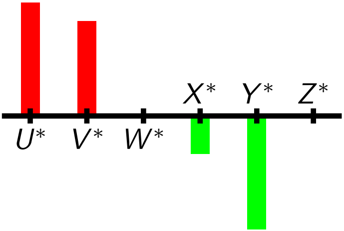
**Theorem**

*Assume $\lambda = (1+\sqrt{5})/2$. Conditional on the bracketing processes $V$ and $Y$ remaining nonzero, $(\widehat{W}_n, \widehat{X}_n)$ converges in distribution to the split Brownian motion*

$$(W^*, X^*) = 2\sqrt{\lambda}(\max\{B^*, 0\}, \min\{B^*, 0\})$$

*where $B^*$ is a standard one-dimensional Brownian motion.*

# Split Brownian motion



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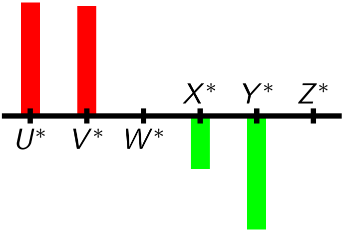

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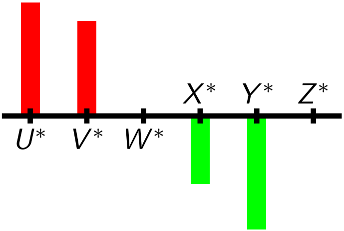

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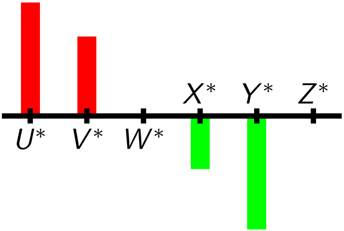

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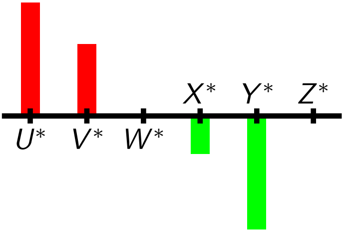

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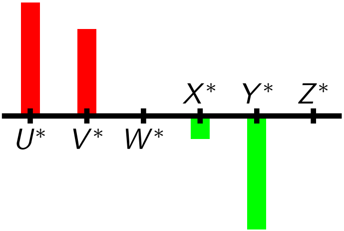

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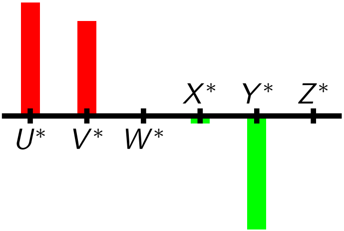

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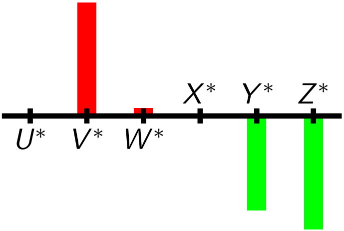

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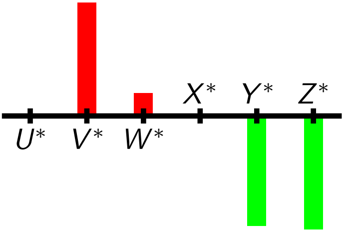

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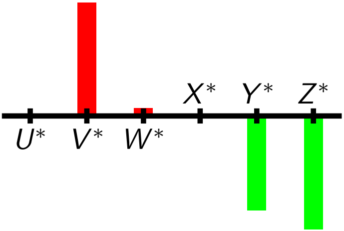

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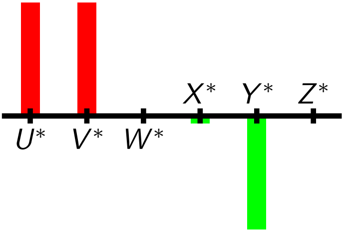

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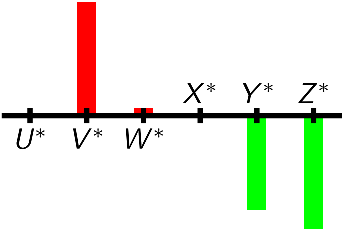

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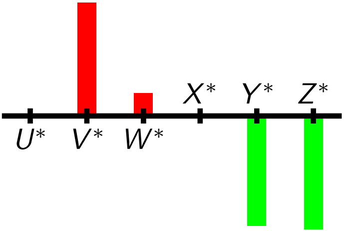

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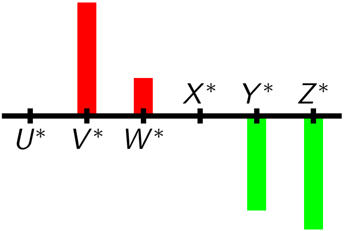

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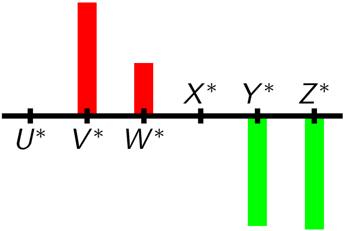

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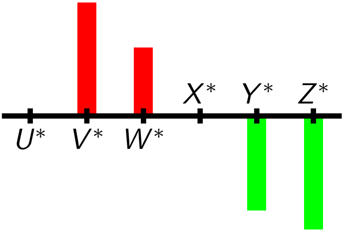
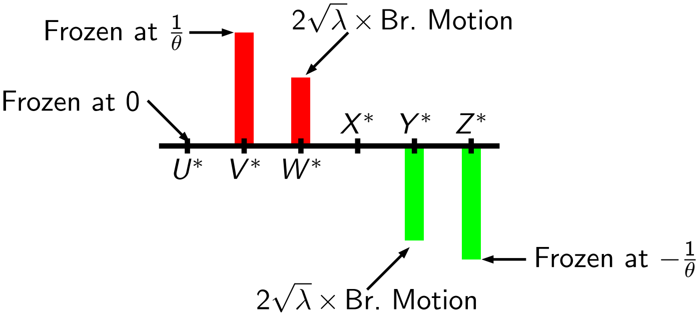


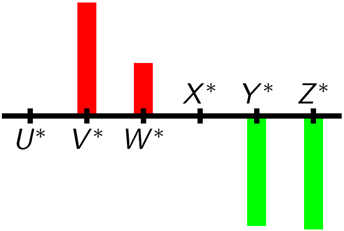
# The other queues




$$\frac{d}{dt}\langle W^*, W^*\rangle = 4\lambda, \qquad \frac{d}{dt}\langle Y^*, Y^*\rangle = 4\lambda$$

$$\frac{d}{dt}\langle W^*, Y^*\rangle = 4$$

# Split Brownian motion



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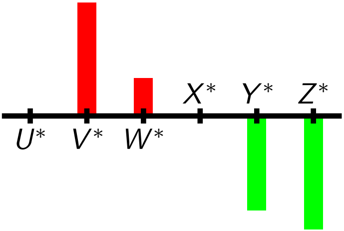

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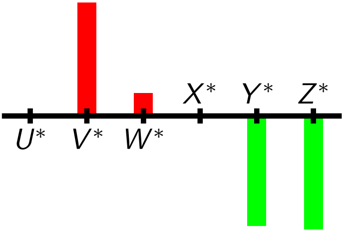

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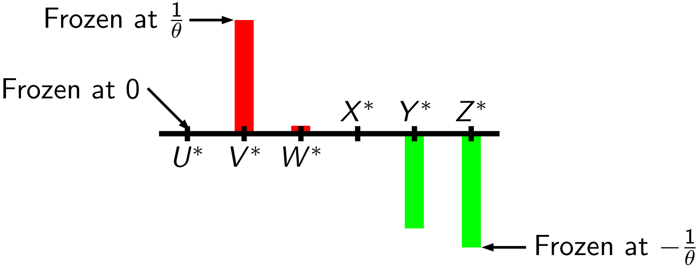

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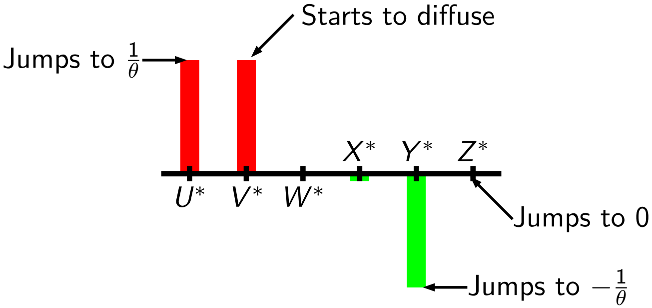

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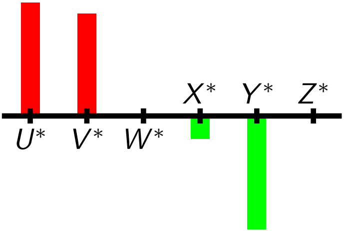
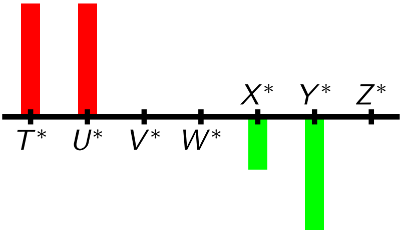



$V^*$ and $X^*$are in a race to zero.

# Split Brownian motion

Suppose $V^*$ wins.



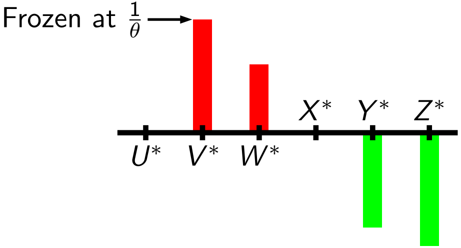

- Reset the "bracketing processes" to be $U^*$ and $X^*$.
- $(V^*, W^*)$ begins executing a split Brownian motion.

# Snapped Brownian motion

Let's consider the $V^*$ process in more detail.
As long as the "bracketing processes" $V^*$ and $Y^*$ remain nonzero, $(W^*, X^*)$ executes a split Brownian motion:

$$(W^*, X^*) = 2\sqrt{\lambda}(\max\{B^*, 0\}, \min\{B^*, 0\}),$$

where $B^*$ is a standard one-dimensional Brownian motion.

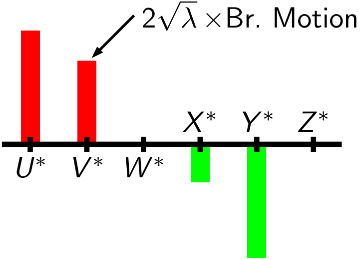


# Snapped Brownian motion

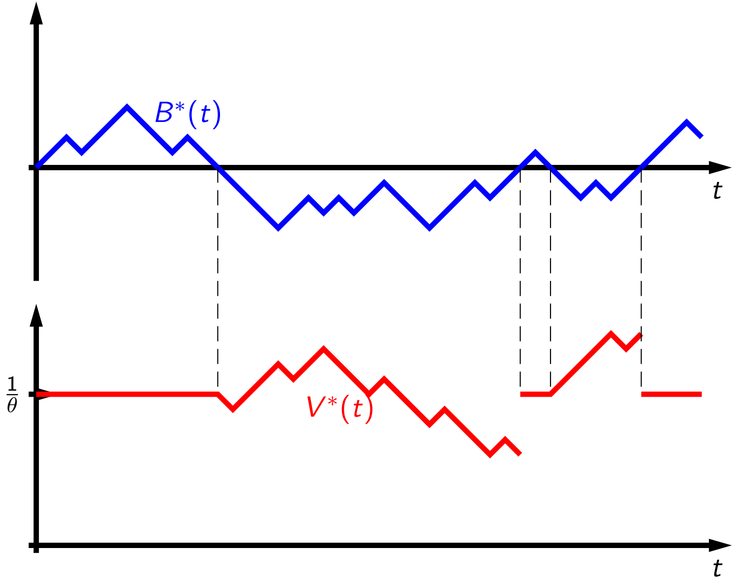
Still have the split Brownian motion,

$$(W^*, X^*) = 2\sqrt{\lambda}(\max\{B^*, 0\}, \min\{B^*, 0\}),$$

but now $V^*$ is diffusing.



# Snapped Brownian motion

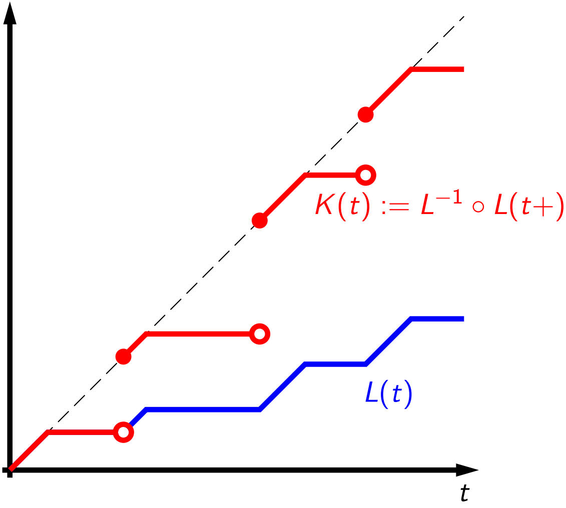


# Properties of snapped Brownian motion $V^*$

- $V^*$ can be constructed using inverse Brownian local time (Tom Kurtz).
- $V^*$ has infinitely many jumps immediately following each time $B^*$ hits zero.
- Jump times and jump sizes of $V^*$ are predictable.
- $V^*$ can be represented using a Poisson random measure and the representation of $B^*$ in terms of its excursions away from zero.
- The absolute values of the jumps are not summable — $V^*$ is not a semimartingale.
- Squares of the jumps are summable.

# Construction of $V^*$ using inverse local time

$L(t)$ is local time of $B^*(t)$ at zero.
$L^{-1}(s) = \min\{t \geq 0 : L(t) \geq s\}$.



# Construction of $V^*$ using inverse local time

Let $C^*$ be another Brownian motion with

$$\frac{d}{dt}\langle C^*, B^* \rangle = \frac{1}{\lambda}.$$

Consider

$$C^* - C^* \circ K + \frac{1}{\theta}.$$

- When $B^*(t) = 0$, we have $K(t) = t$, so

$$C^*(t) - C^* \circ K(t) + \frac{1}{\theta} = \frac{1}{\theta}.$$

- When $B^*$ is on an excursion away from zero, $C^*$ diffuses but $C^* \circ K$ is constant, so

$$C^* - C^* \circ K + \frac{1}{\theta}$$

  diffuses.

This is *almost* what we want.

# Construction of $V^*$ using inverse local time

Occupation time of left half-line:

$$\Gamma(t) = \int_0^t \mathbb{I}_{\{B^*(u)<0\}}\, du.$$

Snapped Brownian motion:

$$V^* = 2\sqrt{\lambda}\Big[C^* \circ \Gamma - \ C^* \circ \Gamma \circ K\Big] + \frac{1}{\theta}.$$

- When $B^*(t) = 0$, $K^*(t) = t$ and $V^*(t) = \frac{1}{\theta}$.
- On positive excursions of $B^*$, $\Gamma$ is constant and $V^* = \frac{1}{\theta}$.
- On negative excursions of $B^*$, $C^* \circ \Gamma$ diffuses and $C^* \circ \Gamma \circ K$ is constant because $K$ is constant. $V^*$ diffuses.

# Summary of properties of the limiting model

- At almost every time, there is a two-tick spread.
- The queues at the bid and ask form a two-dimensional correlated Brownian motion.
- The queues behind the bid and ask are frozen at $\frac{1}{\theta}$ and $-\frac{1}{\theta}$.
- When the queue at the bid or the ask is depleted, we have a three-tick spread.
- We transition through the three-tick spread using the concept of a snapped Brownian motion.

# Now what?

- Complete the proofs so we in fact have a methodology for diffusion limits of limit-order books – summer project.
- Remove the Poisson assumptions.
- Permit orders to have random size.
- Understand what happens if $\lambda \neq \frac{1}{2}(1+\sqrt{5})$.
- Expand the range of prices where orders can arrive. Get a model that has diffusive queues behind the bid and ask and agrees with empirical observations.
- Provide more flexibility to the cancellation, e.g., cancellation to avoid being "picked off."
- Consider optimal execution in the limiting model.
- Consider market making in the limiting model.

THANK YOU FOR YOUR ATTENTION.