Model Free Hedging

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Warwick Statistics

SF@W

Overview

- The classical model-based approach
- Robust or model-independent pricing and hedging
- Primal and dual versions of the problem
- Martingale inequalities
- The Skorokhod embedding problem
- Martingale optimal transport
- PCOCs and fake diffusions

A partial and incomplete list of contributors to the subject

Acciaio, Beiglböck, Bouchard, Brown, Carr, Cox, Davis, Dolinsky, Dupire, Henry-Labordère, Huang, Kahale, Kardaras, Klimmek, Laurance, Lee, Oberhauser, Obloj, Neuberger, Nutz, Penkner, Rogers, Schachermayer, Spoida, Soner, Tan, Temme, Touzi, Tsuzuki, Wang, Wang.

The classical approach to derivative pricing

The standard approach is to postulate a model $\mathcal{M} = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, S = (S_t)_{t \geq 0})$ (or a parametric family of models $\mathbb{M} = \{\mathcal{M}_\lambda; \lambda \in \Lambda\}$).

Then (working in a discounted universe) we price a contingent claim $F_T = F(S_t; 0 \leq t \leq T)$ as a expectation under an equivalent martingale measure $\mathbb{Q}$:

$$\mathbb{E}^{\mathbb{Q}}[F_T]$$

Example

The Black-Scholes-Merton model

For the family of exponential Brownian motion models
$dS_t / S_t = \mu dt + \sigma dW_t$,
for a call with payoff $(S_T - K)^+$, the price is C_{BS} where

$$C_{BS} = C_{BS}(T, K; 0, S_0; \mu, \sigma)$$

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$$C_{BS} = C_{BS}(T, K; 0, S_0; \mu, \sigma)$$

Underpinning the theory is the notion of complete markets and replication:

$$F_T = \mathbb{E}^{\mathbb{Q}}[F_T] + \int_0^T \theta_t dS_t \qquad \mathbb{P}/\mathbb{Q} \text{ a.s.}$$

In a complete market the price and hedge are uniquely specified and replication is perfect provided the model is a perfect description of the real world.

For convex payoffs (eg puts and calls) $C_{BS}(\sigma)$ is increasing in σ. Hence, in practice the Black-Scholes-Merton model is calibrated using a liquid option.

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A reverse approach: constructing models to match vanilla option prices

Typically we are not given $\mathcal{M} = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, S)$ but rather we observe a collection of prices for traded securities, including derivatives.

Rather than starting with the model the issue is to find a model $\mathcal{M}$ to match the prices of traded assets.

In the base case we assume that in addition to the asset itself, vanilla puts and calls are liquidly traded and are used as inputs.

Other derivatives are treated as exotics.

Lemma

Suppose, for fixed T, call prices are known for every strike $K \in (0,\infty)$. Then, assuming

$$C(T,K) = \mathbb{E}^{\mathbb{Q}}[(S_T - K)^+]$$

we have

$$\mathbb{Q}(S_T > K) = \frac{\partial}{\partial K} C(T,K) \qquad \mathbb{Q}(S_T \in dK) = \frac{\partial^2}{\partial^2 K} C(T,K)$$

Corollary (Breeden and Litzenberger)

Any co-maturing European claim can be priced and hedged perfectly.

$$\begin{aligned}
f(y) &= f(x) + (y-x)f'(x) + \int_x^\infty f''(k)(y-k)^+ dk + \int_0^x f''(k)(k-y)^+ dk \\
F(S_T) &= F(S_0) + (S_T - S_0)F'(S_0) + \int_{S_0}^\infty F''(k)(s-k)^+ dk + \int_0^{S_0} F''(k)(k-s)^+ dk \\
\mathbb{E}^{\mathbb{Q}}[F(S_T)] &= F(S_0) + 0 + \int_{S_0}^\infty F''(k)C(T,k)dk + \int_0^{S_0} F''(k)P(T,k)dk
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Corollary (Neuberger - Dupire)

Given the prices of vanilla puts and calls, and assuming that the underlying is continuous semi-martingale the the fixed leg of a T-maturity variance swap is model independent and given by

$$\int_{S_0}^{\infty} \frac{2C(T,k)}{k^2} dk + \int_0^{S_0} \frac{2P(T,k)}{k^2} dk$$

Proof.

$$d(-2 \ln S_t) = -2 \frac{dS_t}{S_t} + \frac{d[S]_t}{S_t^2}$$

Then if $dS_t = S_t \sigma_t dW_t$,

$$\int_0^T \sigma_t^2 dt = \int_0^T \frac{d[S]_t}{S_t^2} = \int_0^T \frac{2}{S_t} dS_t - 2 \ln(S_T/S_0)$$

$$= \int_0^T \left[\frac{2}{S_t} - \frac{2}{S_0} \right] dS_t + \int_{S_0}^{\infty} \frac{2}{k^2}(S_T - k)^+ dk + \int_0^{S_0} \frac{2}{k^2}(k - S_T)^+ dk$$

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$$\begin{aligned} \int_0^T \sigma_t^2 dt = \int_0^T \frac{d[S]_t}{S_t^2} &= \int_0^T \frac{2}{S_t} dS_t - 2\ln(S_T/S_0) \\ &= \int_0^T \left[\frac{2}{S_t} - \frac{2}{S_0}\right] dS_t + \int_{S_0}^{\infty} \frac{2}{k^2}(S_T - k)^+ dk + \int_0^{S_0} \frac{2}{k^2}(k - S_T)^+ dk \end{aligned}$$

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Model free Hedging

Suppose the underlying is a forward price (ie suppose we are working in discounted units).

Suppose calls and puts of a single fixed maturity are liquidly traded.

The objective is to price and hedge a co-maturing (path-dependent) exotic option.

Definition

A perfectly calibrated model is a quintuple $(\Omega, \mathcal{F}, \mathbb{Q}, \mathbb{F}, S)$ such that S is a $(\mathbb{Q}, \mathbb{F})$-martingale and $\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+] = C(T, K)$ for each traded vanilla call.

More generally, given a set of option prices

- is there a perfectly calibrated model consistent with those option prices?
  - is there a model with continuous paths?
  - is there a model with paths of finite (quadratic) variation?
- if there is, is it unique? If not
  - one ambition is to choose a parsimonious model which captures essential features of the problem is realistic
  - another ambition is to search for the class $\mathbb{M}$ of all consistent models
- if there is not, is there an arbitrage?
  - is there a model-independent (strong) arbitrage?
  - is there a model-dependent (weak) arbitrage??

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A perfectly calibrated model can be used for pricing and hedging.

Classical approach

{ Parametric Family of Models } $\Rightarrow$ { Prices of vanillas } $\Rightarrow$ { Calibrated models } $\Rightarrow$ { Prices and hedges of exotics }

Model-free approach

{ Prices of underlyings and vanillas } $\Rightarrow$ { Perfectly calibrated Models/ Classes of models } $\Rightarrow$ { Prices and hedges for exotics }

Classical approach

$$\left\{ \begin{array}{c} \text{Parametric} \\ \text{Family} \\ \text{of Models} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \text{Prices of} \\ \text{vanillas} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \text{Calibrated} \\ \text{models} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \text{Prices} \\ \text{and hedges} \\ \text{of exotics} \end{array} \right\}$$

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The pricing (primal) problem

Suppose we are given a family $\{(S_{T_\alpha} - K_\alpha)^+\}_{\alpha \in A}$ of traded calls with associated prices $C(T_\alpha, K_\alpha)_{\alpha \in A}$.

The class $\mathbb{M}$ of perfectly calibrated models is the class of models $\mathcal{M}$ such that $\mathbb{E}^{\mathbb{Q}}[(S_{T_\alpha} - K_\alpha)^+] = C(T_\alpha, K_\alpha)\ \forall \alpha \in A$.

Now consider adding an extra security $F_T = F(((S_t)_{0 \leq t \leq T}))$ (an exotic). Assuming $\mathcal{M}$ is non-empty the primal problem is to find:

$$\mathcal{P} = \sup_{\mathcal{M} \in \mathbb{M}} \mathbb{E}^{\mathbb{Q}}[F_T]$$

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The dual (hedging) problem

In the setting as above, suppose we can find $\pi = \{(\pi_i)_{i=1,\dots,N}\}, \alpha = \{(\alpha_i)_{1 \le i \le N}\}, \theta = (\theta_t)_{0 \le t \le T}$ such that (π, α, θ) is a semi-static superhedge, i.e.

$$F_T \le \sum_{i=1}^{N} \pi_i (S_{T_{\alpha_i}} - K_{\alpha_i})^+ + \int_0^T \theta_t dS_t$$

Remarks:

We want to be able to give a pathwise interpretation to the integral. For this we may need to decide which paths of S are to be considered as feasible (càdlàg? continuous?), and to put restrictions on θ, (elementary? bounded variation?)

Then the price of the super-hedging portfolio is

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The dual problem is to find

$$\mathcal{D} = \inf \sum_{i=1}^{N} \pi_i C(T_{\alpha_i}, K_{\alpha_i})$$

where the infimum is taken over semi-static superhedges.

Taking expectations in any perfectly calibrated model and for any semi-static superhedge

$$\mathbb{E}^{\mathbb{Q}} F_T \leq \sum_{i=1}^{N} \pi_i C(T_{\alpha_i}, K_{\alpha_i})$$

we get weak duality

$$\mathcal{P} \leq \mathcal{D}$$

For specific examples, and under some general assumptions, there is strong duality $\mathcal{P} = \mathcal{D}$.

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Martingale inequalities

Let $(x_0, x_1, \dots x_n)$ be a sequence of real numbers.

Let $f = f(x_0, x_1, \dots, x_n)$ be given.

Suppose we can find $(c_i, h_i)_{0 \le i \le n}$ where $c_i = c_i(x_i)$ and $h_i = h_i(x_0, x_1, \dots, x_i)$ such that for all sequences $(x_0, x_1, \dots, x_n)$

$$f(x_0, x_1, \dots, x_n) \le \sum_{i=0}^{n} c_i(x_i) + \sum_{i=0}^{n-1} (x_{i+1} - x_i) h_i(x_0, x_1, \dots, x_i)$$

Then if $M = (M_k)_{0 \le k \le n}$ is a discrete-parameter martingale then

$$f(M_0, M_1 \dots, M_n) \le \sum_{i=0}^{n} c(M_i) + \sum_{i=0}^{n-1} (M_{i+1} - M_i) h_i(M_0, M_1, \dots, M_i)$$

and then

$$\mathbb{E}[f(M_0, M_1 \dots, M_n)] \le \sum_{i=0}^{n} \mathbb{E}[c(M_i)]$$

This inequality relates the path-dependent functional to the marginals.

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For any sequence $(x_0, x_1, \dots x_n)$, let $y_k = \max\{x_0, x_1, \dots x_k\}$. Then

$$\sum_{i=0}^{n-1} h(y_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} h(y_i)(y_{i+1} - y_i) + h(y_n)(x_n - y_n)$$

Then with $h(x) = 4x$, and using $4u(v - u) \leq 2(v^2 - u^2)$,

$$\begin{aligned} \sum_{i=0}^{n-1} 4y_i(x_{i+1} - x_i) &= \sum_{i=0}^{n-1} 4y_i(y_{i+1} - y_i) + 4y_n(x_n - y_n) \\ &\leq \sum_{i=0}^{n-1} 2(y_{i+1}^2 - y_i^2) + 4y_n(x_n - y_n) = -2y_n^2 - 2y_0^2 + 4x_n y_n \end{aligned}$$

and then

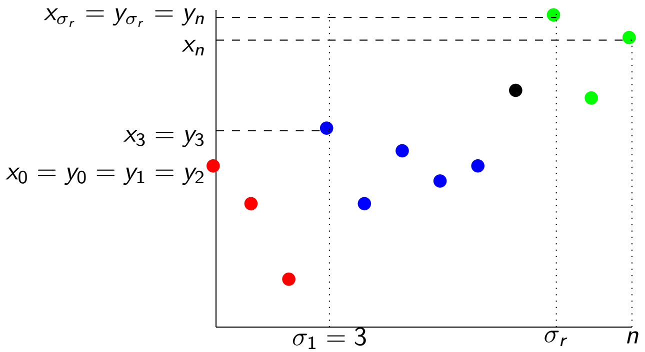
$$\sum_{i=0}^{n-1} 4y_i(x_{i+1} - x_i) + 2x_0^2 + y_n^2 - 4x_n^2 \leq 4x_n y_n - y_n^2 - 4x_n^2 \leq 0$$

For any sequence

$$y_n^2 \leq 4x_n^2 - 2x_0^2 + \sum_{i=0}^{n-1} 4y_i(x_{i+1} - x_i)$$

and it follows that, for any martingale

$$\mathbb{E}\left[(\sup_{0 \leq k \leq n} M_k)^2 \right] \leq 4\mathbb{E}[M_n^2] - 2\mathbb{E}[M_0^2]$$



$$\begin{aligned}
\sum_{i=0}^{n-1} h(y_i)(x_{i+1} - x_i) &= \sum_{s=0}^{r-1} \sum_{i=\sigma_i}^{\sigma_{i+1}-1} h(y_i)(x_{i+1} - x_i) + \sum_{i=\sigma_r}^{n-1} h(y_i)(x_{i+1} - x_i) \\
&= \sum_{s=0}^{r-1} h(y_s)(y_{s+1} - y_s) + h(y_n)(x_n - y_n) \\
&= \sum_{i=0}^{n-1} h(y_i)(y_{i+1} - y_i) + h(y_n)(x_n - y_n)
\end{aligned}$$

One-touch digitals

Suppose $S_0 < B$ and consider the first time H_B that S gets to level B or above. Consider a one-touch digital which pays a unit amount on this event.

Then, for $K < B$

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\begin{aligned}
I\{\max_{0 \le u \le T} S_u \ge B\} &\le \frac{1}{B-K}(S_T - K)^+ + \int_0^T I\{t \ge H_B\} \frac{-1}{(B-K)} dS_t \\
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Then

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\sup_{\mathcal{M} \in \mathbb{M}} \mathbb{Q}\{\max_{0 \le u \le T} S_u \ge B\} = \mathcal{P} \le \mathcal{D} = \inf_{K < B} \frac{C(T,K)}{B-K}
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Note $C(T,K)$ is a decreasing convex function with $C(0) = S_0 < B$, and we can find the infimum = minimum over K by calculus, or by picture.

We can prove equality throughout by finding a perfectly calibrated model, and a K^* for which $\mathbb{Q}\{\max_{0 \le u \le T} S_u \ge B\} = C(T,K^*)/(B-K^*)$.

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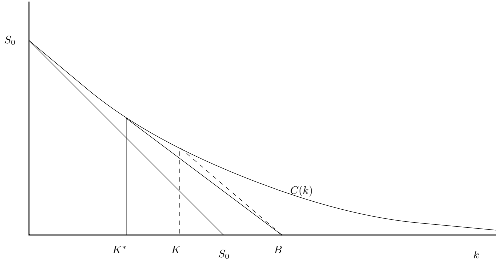
S_0
$C(k)$
K^*
K
S_0
B
k

Skorokhod Embedding Problems

Definition

Let W be Brownian motion started at 0. Let η be a centred probability measure. The Skorokhod embedding problem (SEP) is to find a stopping time τ such that $W_\tau \sim \eta$.

We can switch to $W_0 = S_0$ and η has mean S_0 by a translation.

In the base problem, knowledge of a continuum of calls for a single maturity T is equivalent to knowledge of the law of M_T. But, by a theorem of Monroe, any right continuous martingale can be written as a time-change $M_t = W_{\tau_t}$ of Brownian motion.

Conversely, given W a solution of the SEP such that $W_\tau \sim \mu$ we can find a perfectly calibrated model by setting $S_t = W_{(t/T-t)\wedge\tau}$.

If F is invariant under time change, so that $F((S_t)_{0\le t\le T}) = F((W_t)_{0\le t\le \tau})$ we have

$$\sup_{\mathcal{M}\in\mathbb{M}} \mathbb{E}[F] = \sup_{\tau: W_\tau\sim\mu} \mathbb{E}[F((W_t)_{0\le t\le\tau})]$$

and the search over models becomes a search over solutions of the SEP.

For increasing functions ϕ, the solution of the SEP which maximises $\mathbb{E}[\phi(\sup_{u\le\tau} W_s)]$ is the Azéma-Yor solution.

Hence we can prove optimality (and no duality gap) for the one-touch digital.

Renewed interest in the Skorokhod Embedding Problem:

- New proofs of existing results based on pathwise inequalities
- New (financial) interpretations of existing results
- New embeddings based on maximising functionals
- Forward starting versions (non-trivial initial law) and non-centred versions

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Optimal transport

Let μ, ν be probability measures on a space E and let $c : E \times E \mapsto \mathbb{R}$ be a cost function.

The optimal transport problem, (Monge, Kantorovich, ..., Villani ...) is to find

$$\inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[c(X,Y)]$$

If $E = \mathbb{R}$ and $c(x,y) = \gamma(y-x)$ with γ convex, and if μ is atom free, then the solution is trivial; set $Y = F_\nu^{-1}(F_\mu(X))$.

Suppose there exists ϕ, ψ s.t. $c(x,y) \geq \phi(x) + \psi(y)$. Then for *any* X, Y

$$\mathcal{P}(\mu,\nu) = \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}[c(X,Y)] \geq \sup_{\phi,\psi : c \geq \phi + \psi} \mathbb{E}[\phi(X) + \psi(Y)] = \mathcal{D}(\mu,\nu)$$

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Martingale Optimal Transport

Suppose $E = \mathbb{R}$ and we add a condition that $\mathbb{E}[Y|X] = X$ to represent a martingale condition.

The martingale optimal transport problem is to find

$$\mathcal{P}_{mg} = \inf_{X \sim \mu, Y \sim \nu, \mathbb{E}[Y|X]=X} \mathbb{E}[c(X,Y)]$$

Suppose there exists ϕ, ψ, h such that
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where ρ is a joint law of (X,Y) and $\mathcal{A} = \mathcal{A}(\mu,\nu) = \{\rho : \int_y \rho(dx,dy) \sim \mu(dx); \int_x \rho(dx,dy) \sim \nu(dy); \int_y (y-x)\rho(dx,dy) = 0\}$, and the supremum is taken over ϕ, ψ, h such that $c(x,y) \geq \phi(x) + \psi(y) + h(x)(y-x)$.

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An Example

Suppose $\mu \sim U[-1,1]$ and $\nu \sim U[-2,2]$. Suppose $c(x,y) = -|y-x|$.

Primal problem:

Let $Z = \pm 1$ with $\mathbb{P}(Z=1) = \frac{1}{2} = \mathbb{P}(Z=-1)$. Let $Y = X + Z$. Then, if $X \sim \mu$ we have $Y \sim \nu$ and $\mathbb{E}[Y|X] = X$.

Then (X,Y) is a solution for the problem, and in this case $\mathbb{E}[-|Y-X|] = -1$. Hence $\mathcal{P} \leq -1$.

Dual problem:

Since $-|b| \geq -\frac{1}{2}(b^2+1)$ and $(y-x)^2 = y^2 - x^2 - 2x(y-x)$,

$$-|Y-X| \geq -\frac{Y^2}{2} + \frac{X^2}{2} + X(Y-X) - \frac{1}{2}$$

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Finally $-1 \geq \mathcal{P} \geq \mathcal{D} \geq -1$ and there is equality throughout.

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Martingale Optimal Transport: Open Issues

- Solve the problem for explicit cost functionals eg $c(x, y) = |y - x|$ or $c(x, y) = (\ln y - \ln x)^2$.
- Characterisations of the optimal solution (eg analogues of Brenier's theorem).
- Prove duality in a general setting.
- Extend to multiple time-steps by concatenation.

PCOCs and fake diffusions

Definition

A fake Brownian motion is a martingale $X = (X_t)_{t \geq 0}$ with $N(0, t)$ marginals which is **not** Brownian motion.

(Note (X_s, X_t) is not jointly normal.) There are several constructions of fake Brownian motion, typically involving dropping either the path-continuity property, or the Markov property.

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Definition

A PCOC is a *Processus Croissant d'Ordre Convexe*: given a set of marginals which is increasing in convex order, a PCOC is a martingale which matches those marginals.

The mathematical finance interpretation

- given a set of call option prices $\{C(T, K)\}_{T \geq 0, K \geq 0}$ satisfying no-arbitrage, construct a perfectly calibrated model
  - elegantly
  - to maximise some path-dependent functional
- given an exotic derivative, find the cheapest superhedging strategy.
- duality gap?

Note, the Dupire construction gives a perfectly calibrated model.

Local volatility models

Lemma (Dupire)

Suppose call prices are known for every strike $K \in (0, \infty)$ and every $T \in (0, T]$. Assuming $C(T, K)$ is sufficiently differentiable, there exists a unique diffusion of the form

$$dS_t = S_t \sigma(t, S_t) dB_t$$

such that

$$C(T, K) = \mathbb{E}[(S_T - K)^+]$$

In particular $\sigma(t, s)$ solves

$$\sigma(T, K)^2 = \frac{2 \frac{\partial}{\partial T} C(T, K)}{K^2 \frac{\partial^2}{\partial^2 K} C(T, K)}$$

Are there other models which calibrate perfectly to the data? If so, why choose the local-vol model?

Summary and conclusions

The model-free framework provides an alternative to the classical approach.

The idea is to use liquidly traded instruments (including vanilla derivatives) to restrict attention to the class of perfectly calibrated models (primal approach) or as hedging instruments (dual approach).

Implications for trading of financial derivatives:
Upper and lower price bounds; sub and superhedges.
But bounds are often wide.

Connections with mathematics:
Martingale inequalities, Skorokhod embeddings, martingale optimal transport, fake diffusions, ...

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Thankyou for your attention!

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