Valuation in illiquid markets

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Valuation of Securities

Liquid markets → law of one price $E^Q[X]$

Illiquid markets
↗ ask price (upper price)
↘ bid price (lower price)

Example for a Two Price Evaluation

Consider a public debt obligation

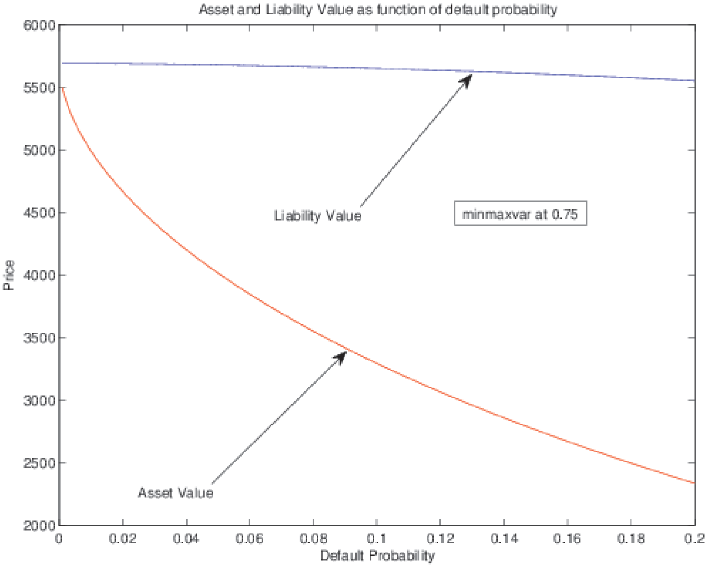
Possibility of default or other causes of illiquidity:
The lender (investor) has to discount the value of the loan or bond
$\longrightarrow$ bid price

The borrower (issuer of debt) does not contemplate default:
For him the obligation is risk free $\longrightarrow$ ask price

At maturity: convergence of bid and ask

Consequences:
- Bid price will vary with the changes in the issuer's credit status
- Ask price will remain relatively steady

Further fact: Value depends on the direction of the trade


Asset and Liability Value as function of default probability
6000
5500
5000
4500
4000
3500
3000
2500
2000
Price
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
Default Probability
Liability Value
minmaxvar at 0.75
Asset Value

Acceptability of Cashflows

Outcome (cashflow) of a risky position: X random variable

In perfectly liquid markets: pricing kernel given by a risk-neutral measure Q

value of the position: $E^Q[X]$

position is acceptable if: $E^Q[X] \geq 0$

Real markets:

Instead of a unique probability measure Q we have to consider a set of probability measures (scenarios) $Q \in \mathcal{M}$

$$E^Q[X] \geq 0 \quad \text{for all } Q \in \mathcal{M} \qquad \text{or} \qquad \inf_{Q \in \mathcal{M}} E^Q[X] \geq 0$$

Coherent Risk Measures

Specification of $\mathcal{M}$ (test measures, generalized scenarios)

Axiomatic theory of risk measures: desirable properties

Monotonicity: $X \geq Y \Longrightarrow \varrho(X) \leq \varrho(Y)$

Cash invariance: $\varrho(X + c) = \varrho(X) - c$

Scale invariance: $\varrho(\lambda X) = \lambda \varrho(X), \quad \lambda \geq 0$

Subadditivity: $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$

Examples: Value at Risk (VaR) (not coherent)
Tail-VaR (expected shortfall)

Any coherent risk measure has a representation

$$\varrho(X) = - \inf_{Q \in \mathcal{M}} E^{Q}[X]$$

Operationalization

Link between acceptability and concave distortions

Assume the set $\mathcal{M}$ is convex and the operator

$$\varrho(X) = -\inf_{Q\in\mathcal{M}} E^Q[X] = \sup_{Q\in\mathcal{M}} E^Q[-X]$$

is law invariant and comonotone (i.e. a spectral risk measure)

$\Rightarrow \exists$ concave distortion Ψ (i.e. a concave distribution function on $[0,1]$) s.t.

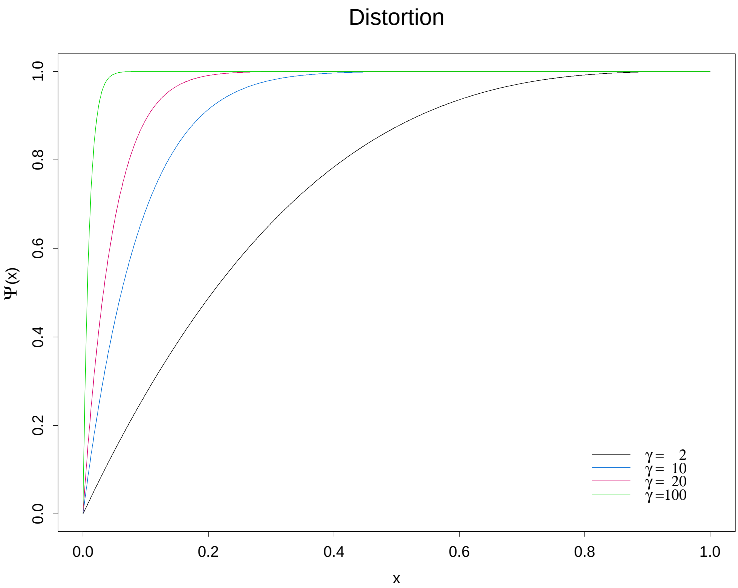
$$\varrho(X) = -\int_{-\infty}^{+\infty} x d\Psi(F(x))$$

Acceptability means then $\int_{-\infty}^{+\infty} x d\Psi(F(x)) \geq 0.$

The corresponding set of probability measures (the supporting set) is given by

$$\mathcal{M} = \{Q \in \mathcal{P} \mid \widehat{\Psi}(P(A)) \leq Q(A) \leq \Psi(P(A)) \;\; (A \in \mathfrak{A})\}$$

where $\widehat{\Psi}(x) := 1 - \Psi(1-x)$


Distortion
Ψ(x)
0.0
0.2
0.4
0.6
0.8
1.0
x
γ = 2
γ = 10
γ = 20
γ =100

Families of Distortions (1)

Consider families of distortions $(\Psi^\gamma)_{\gamma \geq 0}$

γ stress level

Example: MIN VaR

$$\Psi^\gamma(x) = 1 - (1 - x)^{1+\gamma} \qquad (0 \leq x \leq 1, \gamma \geq 0)$$

Statistical interpretation:

Let γ be an integer, then $\varrho_\gamma(X) = -E(Y)$ where

$$Y \overset{\text{law}}{=} \min\{X_1, \dots, X_{\gamma+1}\}$$

and $X_1, \dots, X_{\gamma+1}$ are independent draws of X

Families of Distortions (2)

Further examples: MAX VaR

$$\Psi^{\gamma}(x) = x^{\frac{1}{1+\gamma}} \quad (0 \leq x \leq 1, \gamma \geq 0)$$

Statistical interpretation: $\varrho_{\gamma}(X) = -E[Y]$

where Y is a random variable s.t.

$$\max\{Y_1, \ldots, Y_{\gamma+1}\} \stackrel{\text{law}}{=} X$$

and $Y_1, \ldots, Y_{\gamma+1}$ are independent draws of Y.

Combining MIN VaR and MAX VaR: MAX MIN VaR

$$\Psi^{\gamma}(x) = (1-(1-x)^{1+\gamma})^{\frac{1}{1+\gamma}} \qquad (0 \leq x \leq 1, \gamma \geq 0)$$

Interpretation: $\varrho_{\gamma}(X) = -E[Y]$ with Y s.t.

$$\max\{Y_1, \ldots, Y_{\gamma+1}\} \stackrel{\text{law}}{=} \min\{X_1, \ldots, X_{\gamma+1}\}$$

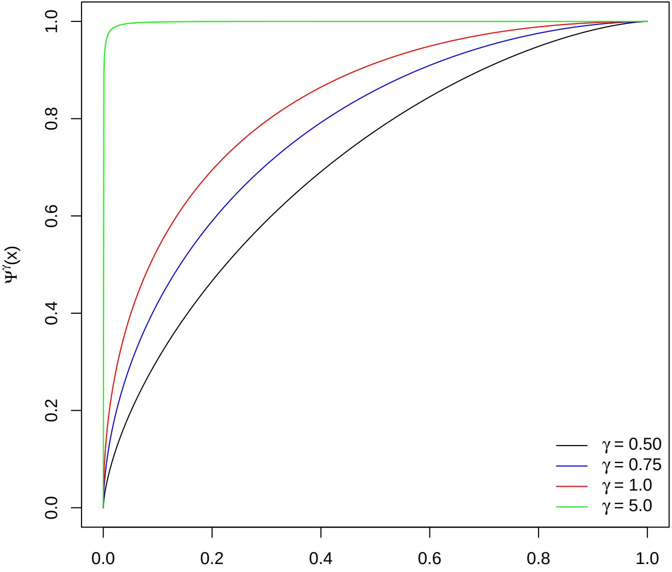
Families of Distortions (3)

Distortion used: MIN MAX VaR

$$\Psi^{\gamma}(x) = 1 - \left(1 - x^{\frac{1}{1+\gamma}}\right)^{1+\gamma} \qquad (0 \leq x \leq 1, \gamma \geq 0)$$

$$\varrho_{\gamma}(X) = -E[Y] \quad \text{with } Y \text{ s.t.} \quad Y \overset{\text{law}}{=} \min\{Z_1, \ldots, Z_{\gamma+1}\},$$
$$\max\{Z_1, \ldots, Z_{\gamma+1}\} \overset{\text{law}}{=} X$$

Families of Distortions (4)



Marking Assets and Liabilities

Assets: Cash flow to be received $X \geq 0$

Largest value $b(X)$ s.t. $X - b(X)$ is acceptable

$$\Rightarrow \quad b(X) = \inf_{Q \in \mathcal{M}} E^Q[X]$$

Bid price or Lower price

Liabilities: Cash flow to be paid out $X \geq 0$

Smallest value $a(X)$ s.t. $a(X) - X$ is acceptable

$$\Rightarrow \quad a(X) = \sup_{Q \in \mathcal{M}} E^Q[X]$$

Ask price or Upper price

Explicit Bid and Ask Pricing

Bid Price of a cash flow X: Acceptability of $X - b(X)$

$$b(X) = \int_{-\infty}^{\infty} x d\Psi(F_X(x))$$

Ask Price of a cash flow X: Acceptability of $a(X) - X$

$$a(X) = -\int_{-\infty}^{\infty} x d\Psi(1 - F_X(-x))$$

Examples: Calls and Puts

$$bC(K,t) = \int_K^{\infty} \big(1 - \Psi(F_{S_t}(x))\big) dx$$

$$aC(K,t) = \int_K^{\infty} \Psi(1 - F_{S_t}(x)) dx$$

$$bP(K,t) = \int_0^{K} \big(1 - \Psi(1 - F_{S_t}(x))\big) dx$$

$$aP(K,t) = \int_0^{K} \Psi(F_{S_t}(x)) dx$$

Feynman–Kac representation

Eberlein, Glau (2013) (to appear in Applied Mathematical Finance)

$(L_t)_{t\geq 0}$ time-inhomogenous Lévy process (PIIAC)

$$E[\exp(i\xi L_t)] = \exp\left(\int_0^t \theta_s(i\xi)ds\right)$$

$$\theta_s(i\xi) = ib_s\xi - \frac{1}{2}\sigma_s\xi^2 + \int_{\mathbb{R}}(e^{i\xi y} - 1 - i\xi h(y))F_s(dy)$$

usual integrability assumptions

$\mathcal{G}_t$ infinitesimal generator

$\mathcal{A}_t = -\mathcal{G}_t$ pseudo differential operator (PDO) with symbol A_t

$$A_t(\xi) = -\theta_t(-i\xi) \qquad (\xi \in \mathbb{R})$$

$$(\mathcal{A}u = \mathcal{F}^{-1}(A\mathcal{F}(u)) \qquad (u \in C_o^\infty))$$

Pricing of derivatives with payoff function g
(e.g. call option $g(x) = (S_0 e^x - K)^+$)

$$\partial_t u + \mathcal{A}_{T-t} u = 0 \qquad (r = 0)$$
$$u(0) = g$$

Stochastic representation of the solution:

$$u(T-t, x) = E[g(L_T - L_t + x)]$$

Solution of the PIDE

$$\begin{aligned} \partial_t u + \mathcal{A}_{T-t} u &= 0 \\ u(0) &= g \end{aligned}$$

Desirable properties in the case of models for finance:

- unbounded domains (domain is the range of the process L)
- initial condition should cover a large range of options (literature: polynomial boundedness, Lipschitz-continuity, ...)
- variational solution (numerical calculation)

$\longrightarrow$ dampening of the payoff g

$$\begin{aligned} L^2_\eta &:= \{u \in L^1_{\text{loc}} \mid x \to u(x)e^{\eta x} \in L^2\} \\ < u, v >_{L^2_\eta} &:= \int_{\mathbb{R}} u(x)\overline{v}(x)e^{2\eta x}dx \end{aligned}$$

Fourier transform with weights

$$\mathcal{F}_\eta(\varphi) := e^{-\eta \cdot}\mathcal{F}(\varphi e^{\eta \cdot}) \quad (\varphi \in L^2_\eta \text{ resp. } \varphi \in S_\eta)$$

Initial condition: $g \in L^2_\eta$

Explicit solution of the PIDE

Transformation into weighted Fourier transforms

$$\mathcal{F}_\eta(\partial_t u) + \mathcal{F}_\eta(\mathcal{A}_{T-t} u) = 0$$
$$\mathcal{F}_\eta\left(L^2_\eta - \lim_{t\downarrow 0} u(t)\right) = \mathcal{F}_\eta(g)$$

Equivalent to ODE

$$\partial_t \mathcal{F}_\eta(u(t))(\xi) + A_{T-t}(\xi - i\eta)\mathcal{F}_\eta(u(t))(\xi) = 0 \quad \text{(a.e. } \xi \in \mathbb{R})$$
$$\mathcal{F}_\eta(u)(t=0) = \mathcal{F}_\eta(g)$$

Solution

$$\mathcal{F}_\eta(u(t))(\xi) = \mathcal{F}_\eta(g)(\xi) \exp\left(-\int_{T-t}^{T} A_s(\xi - i\eta) ds\right)$$

or (weak solution)

$$u(t,x) = \frac{e^{-\eta x}}{2\pi} \int_{\mathbb{R}} e^{-i\xi(x+i\eta)} \mathcal{F}_\eta(g) e^{-\int_{T-t}^{T} A_s(\xi - i\eta) ds} d\xi$$

Stochastic representation

$$u(T-t, x) = E[g(L_T - L_t + x)]$$

Continuous Time Theory

Underlying uncertainty given by a pure jump Lévy process $(X_t)_{0 \le t \le T}$

Specified by: drift term α, Lévy measure $k(y)dy$ $(y \neq 0)$

Example: Variance gamma

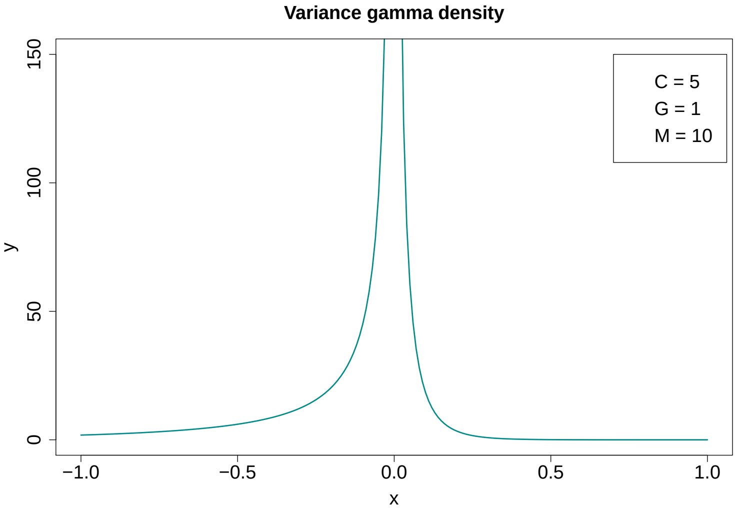
$$k(y) = \frac{C}{|y|}(\exp(-G|y|)1_{\{y<0\}} + \exp(-M|y|)1_{\{y>0\}})$$

Note that

$$\int_{\mathbb{R}} y^2 k(y)dy < \infty$$

Infinitesimal generator $\mathcal{L}$ of the process

$$\mathcal{L}u(x) = \alpha \frac{\partial u}{\partial x}(x) + \int_{\mathbb{R}} \left(u(x+y) - u(x) - \frac{\partial u}{\partial x}(x) y \right) k(y)dy$$


Variance gamma density
C = 5
G = 1
M = 10
y
x
0
50
100
150
−1.0
−0.5
0.0
0.5
1.0


GH Levy process with marginal densities
values of GH (-0.5,100,0,1,0.1) Levy process
100.6
100.4
100.2
100.0
99.8
0.0
0.5
1.0
1.5
2.0
t

Valuation of Financial Contracts

Consider a contract which pays $\phi(X_t)$ at time t

Denote by $u(x,t)$ its time zero value when $X_0 = x$

$\Rightarrow \quad u(x,t) = E\left[e^{-rt}\phi(X_t) \mid X_0 = x\right]$

risk-neutral value for constant interest rate r

Assume that ϕ is such that Feynman–Kac applies
$u(x,t)$ is the solution of the partial integro-differential equation (PIDE)

$$u_t = \mathcal{L}(u) - ru$$

with boundary condition $u(x,0) = \phi(x)$

$\mathcal{G}$-Expectations Using Distortions (1)

Integral part of the PIDE

$$\int_{\mathbb{R}} \underbrace{\frac{(u(x+y,t) - u(x,t) - u_x(x,t)y) \int_{\mathbb{R}} y^2 k(y)dy}{y^2}}_{=:Y_{x,t}} g(y)dy$$

where $g(y) = \dfrac{y^2 k(y)}{\int_{\mathbb{R}} y^2 k(y)dy}$

$\rightarrow$ g is a probability density

Define the distribution function

$$F_{Y_{x,t}}(v) = \int_{A(x,t,v)} g(y)dy$$

where $A(x,t,v) = \{y \mid Y_{x,t} \leq v\}$

$\mathcal{G}$-Expectations Using Distortions (2)

Integral part of the PIDE is now

$$\int_{\mathbb{R}} v dF_{Y_{x,t}}(v)$$

Distorted expectation

$$\int_{\mathbb{R}} v d\Psi(F_{Y_{x,t}}(v))$$

which by decomposition can be written as

$$-\int_{-\infty}^{0} \Psi(P^g(Y_{x,t} \leq v))dv + \int_{0}^{\infty} (1 - \Psi(P^g(Y_{x,t} \leq v)))dv$$

Define the new (distorted) operator

$$\mathcal{G}_{QV} u(x) = \alpha \frac{\partial u}{\partial x}(x) - \int_{-\infty}^{0} \Psi(P^g(Y_{x,t} \leq v))dv + \int_{0}^{\infty} (1 - \Psi(P^g(Y_{x,t} \leq 0)))dv$$

and solve the (distorted) PIDE

$$u_t = \mathcal{G}_{QV}(u) - ru$$

Alternative $\mathcal{G}$-Expectation Approach

Truncation of the Lévy measure

$$\int_{\{|y|\geq\epsilon\}} \big(u(x+y,t)-u(x,t)-u_x(x,t)y\big)k(y)dy$$

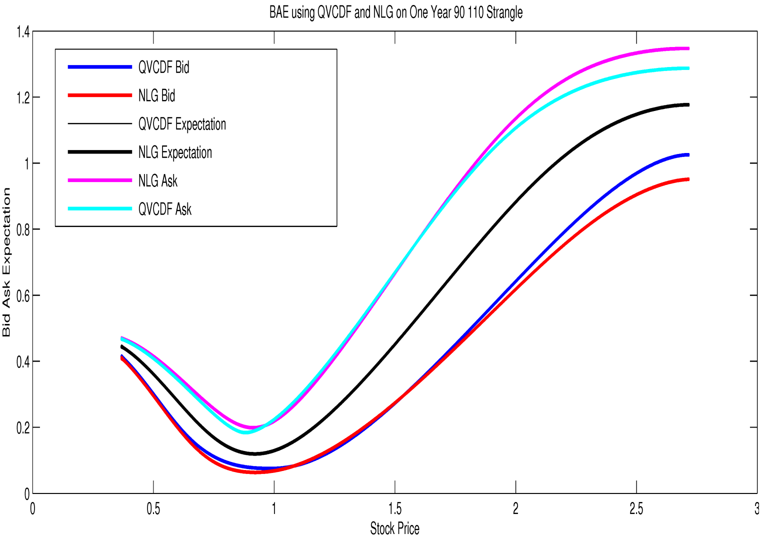
Definition of a probability density $h(y)$ via

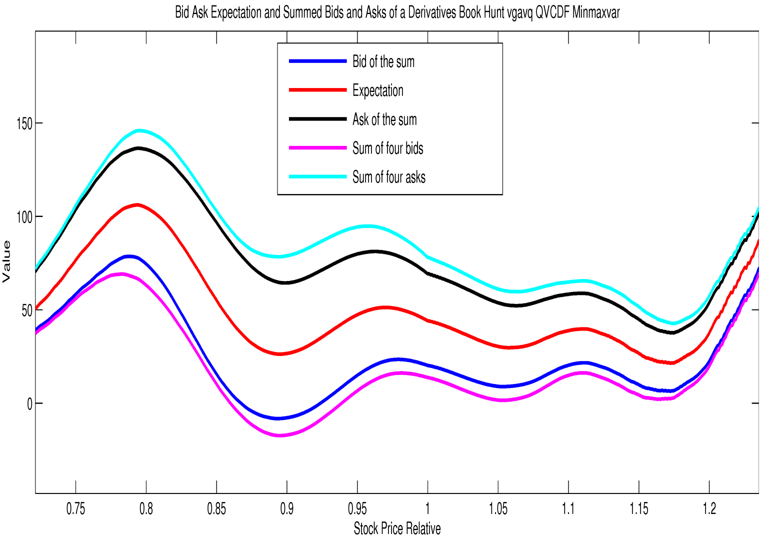
$$\int_{\{|y|\geq\epsilon\}} \underbrace{(u(x+y,t)-u(x,t)-u_x(x,t)y)\left(\int_{\{|y|\geq\epsilon\}} k(y)dy\right)}_{=:\tilde{Y}_{x,t}} h(y)dy$$

where $h(y) = \dfrac{k(y)}{\int_{\{|y|\geq\epsilon\}} k(y)dy} \mathbb{1}_{\{|y|\geq\epsilon\}}$

The distorted operator is now

$$\mathcal{G}_{NL}u(x) = \alpha\frac{\partial u}{\partial x}(x) - \int_{-\infty}^{0} \Psi(P^h(\tilde{Y}_{x,t}\leq v))dv + \int_{0}^{\infty}(1-\Psi(P^h(\tilde{Y}_{x,t}\leq v)))dv$$


BAE using QVCDF and NLG on One Year 90 110 Strangle
QVCDF Bid
NLG Bid
QVCDF Expectation
NLG Expectation
NLG Ask
QVCDF Ask
Bid Ask Expectation
Stock Price
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0.5
1.5
2
2.5
3


Bid Ask Expectation and Summed Bids and Asks of a Derivatives Book Hunt vgavq QVCDF Minmaxvar
Bid of the sum
Expectation
Ask of the sum
Sum of four bids
Sum of four asks
150
100
50
0
Value
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
Stock Price Relative

Securities with very long maturities

The discounted variance gamma model

$\gamma_p(t)$, $\gamma_n(t)$ two independent standard gamma processes

Driving process

$$X(t) = \int_0^t b_p e^{-rs} d\gamma_p(c_p s) - \int_0^t b_n e^{-rs} d\gamma_n(c_n s)$$

$b_p, c_p, b_n, c_n > 0$ scale and shape parameters of the undiscounted gamma processes

Underlying discounted price process

$$M(t) = \exp(X(t) + \omega(t))$$

where $\exp(\omega(t)) = (E[\exp(X(t))])^{-1}$

$\rightarrow$ uniformly integrable martingale with a well-defined limit

$$M(\infty) = \exp(X(\infty) + \omega(\infty))$$

Valuation

Consider now a claim promising for very large T, $F(M(T)) \approx F(M(\infty))$, where the payout is expressed in time zero dollars (F 'nice' function)

Value of the claim at time t

$$w_F(t) = E[F(M(\infty)) \mid \mathcal{F}_t]$$

$\rightarrow$ martingale

Observe now that

$$\begin{aligned} X(\infty) &= X(t) + \int_t^\infty b_p e^{-ru} d\gamma_p(c_p u) - \int_t^\infty b_n e^{-ru} d\gamma_n(c_n u) \\ &\stackrel{(d)}{=} X(t) + e^{-rt} Y \end{aligned}$$

for an independent random variable $Y \sim X(\infty)$

$\Rightarrow$ $$w_F(t) = H(X(t), e^{-rt})$$

Bid and Ask Prices (1)

Martingale condition on $w_F(t)$ (write $v = e^{-rt}$)

$$-rvH_v + \int_{-\infty}^{-\infty} (H(X + y, v) - H(X, v))k(y, v)dy = 0$$

PIDE with boundary condition

$$H(X, 0) = F(\exp(X(\infty) + \omega(\infty)))$$

where

$$k(y, v) = \frac{c_p}{y} \exp\left(-\frac{y}{b_p v}\right) \mathbb{1}_{\{y>0\}} + \frac{c_n}{|y|} \exp\left(-\frac{|y|}{b_n v}\right) \mathbb{1}_{\{y<0\}}$$

Bid and Ask Prices (2)

Rewrite the PIDE

$$rvH_v = \int_{-\infty}^{+\infty} \frac{(H(X+y,v) - H(X,v)) \int_{-\infty}^{+\infty} y^2 k(y,v)dy}{y^2} dF_{QV}(y)$$

where

$$F_{QV}(a) = \frac{1}{\int_{-\infty}^{+\infty} y^2 k(y,v)dy} \int_{-\infty}^{a} y^2 k(y,v)dy$$

Bid price is the solution of the distorted PIDE

$$rvH_v = \int_{-\infty}^{+\infty} \frac{(H(X+y,v) - H(X,v)) \int_{-\infty}^{+\infty} y^2 k(y,v)dy}{y^2} d\Psi^{\gamma}(F_{QV}(y))$$

Ask price: Negative of the bid price of the negative cash flow

Implementation Details

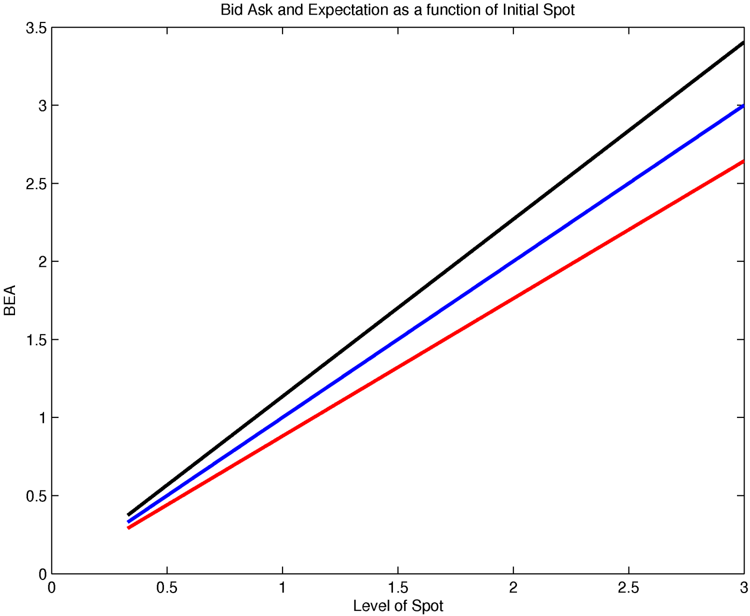
Risk neutral parameters from S & P 500

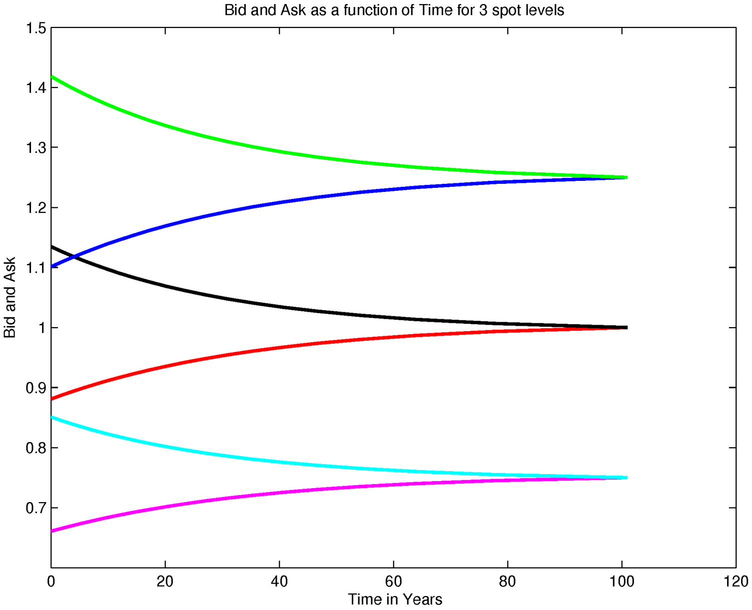
$$r = 0.02966 \qquad b_p = 0.0145 \qquad c_p = 48.4215$$

$$b_n = 0.5707 \qquad c_n = 0.3493$$

Actually solved for a PIDE in $M(t)$:

$$G(M(v), v) = M(v) \exp\left(\omega(\infty) - \omega\left(-\frac{\ln v}{r}\right)\right)\phi_Y(-iv)$$


Bid Ask and Expectation as a function of Initial Spot
3.5
3
2.5
2
1.5
1
0.5
0
BEA
0
0.5
1
1.5
2
2.5
3
Level of Spot


Bid and Ask as a function of Time for 3 spot levels
Bid and Ask
Time in Years
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0
20
40
60
80
100
120

Two Price Valuation of Insurance Losses (1)

Cumulated loss process $L(t)$: e.g. compound Poisson (arrival rate λ)

Loss sizes are iid γ-distributed (scale and shape parameters ζ and κ)

Consider the value process in time zero dollars

$$V(t) = E_t\Big[\int_0^\infty e^{-rs} dL(s)\Big]$$

Let $X(t)$ be the discounted losses to date

$$X(t) = \int_0^t e^{-rs} dL(s)$$

Rewrite

$$\int_0^\infty e^{-rs} dL(s) \;=\; X(t) + e^{-rt}\int_t^\infty e^{-r(s-t)} dL(s) \overset{(d)}{=} X(t) + e^{-rt}Y$$

where Y is an independent copy of $\int_0^\infty e^{-rs} dL(s)$

$\Rightarrow$
$$V(t) = H(X(t), e^{-rt})$$

Two Price Valuation of Insurance Losses (2)

Applying Itô's formula and using the martingale condition (where we replace t by $v = e^{-rt}$)

$$rvH_v = \int_0^{\infty} (H(x+w,v) - H(X,v))k(w,v)dw$$

where $k(w,v)$ is related to the Lévy system for $X(t)$

$$k(w,v) = \frac{\lambda}{\Gamma(\kappa)} \left(\frac{\zeta}{v}\right)^{\kappa} w^{\kappa-1} \exp\left(-\frac{\zeta}{v} w\right)$$

Risk neutral price is the solution of this PIDE

Bid price is the solution of the distorted PIDE

How to distort a measure integral?

Measure Distortions (1)

Consider a possibly infinite measure μ with tails of finite measure and

$$m = \int_{-\infty}^{+\infty} v(y)\mu(dy) < \infty$$

Rewrite this as

$$m = -\int_{-\infty}^{0} \mu(v(y) \leq x)dx + \int_{0}^{\infty} \mu(v(y) > x)dx$$

Distorted measure integrals

$$\mathfrak{m} = -\int_{-\infty}^{0} \Gamma_{+}(\mu(v(y) \leq x))dx + \int_{0}^{\infty} \Gamma_{-}(\mu(v(y) > x))dx$$

for functions $\Gamma_{+}, \Gamma_{-} : \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$

$\Gamma_{\pm}(0) = 0$, monotone increasing, resp. concave and convex, bounded below and above by the identity function

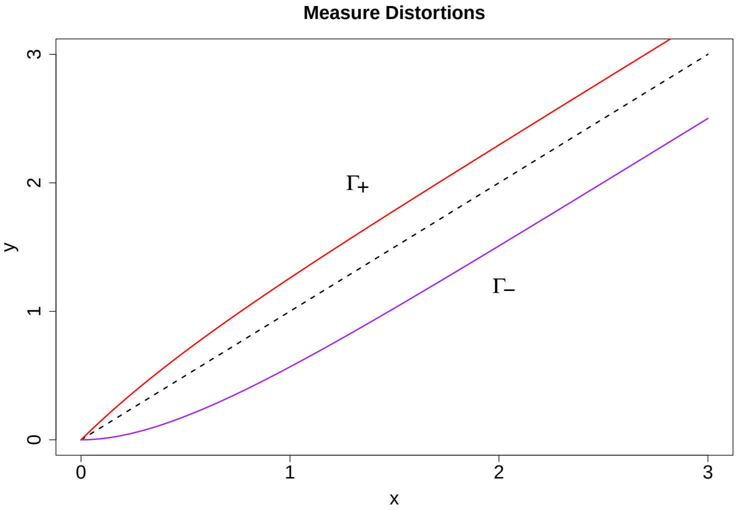
Measure Distortions (2)

Natural Candidates:

$$\Gamma_{+}(x) = x + \alpha(1 - e^{-cx})^{-\frac{1}{1+\gamma_{+}}}$$

$$\Gamma_{-}(x) = x - \frac{\beta}{c}(1 - e^{-c(1+\gamma_{-})x})$$

(Γ_{+} derived from maxvar, Γ_{-} from minvar)


Measure Distortions
Γ+
Γ−
y
x
0
1
2
3
0
1
2
3

Bid Price for the Discounted Cumulated Loss Process

Distorted measure integral for positive variables

$$\mathfrak{m} = \int_0^\infty \Gamma_-(\mu(\chi > x))dx$$

Rewrite this as (integration by parts)

$$\mathfrak{m} = -\int_0^\infty x d\Gamma_-(\mu(\chi > x))$$

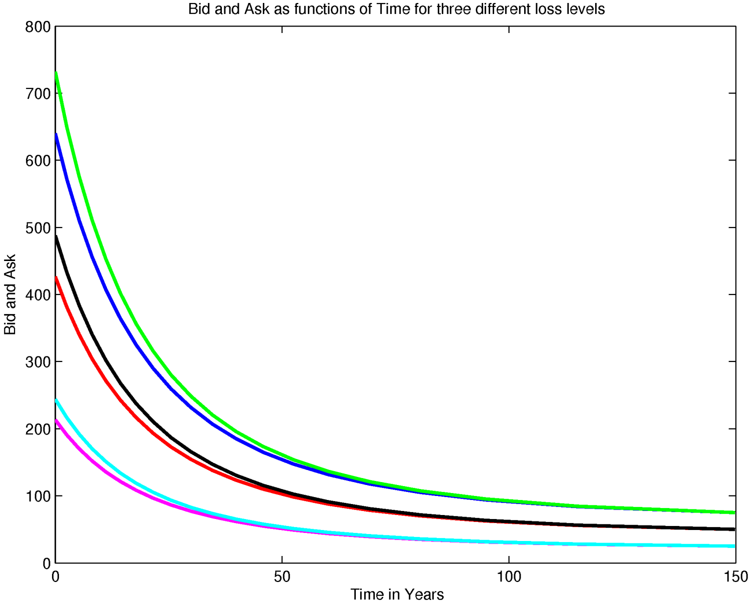
Now choose $\chi(y) = H(X + y, v) - H(X, v), \qquad \mu(dy) = k(y, v)dy$

Bid price is the solution of

$$rvH_v = -\int_0^\infty x d\Gamma_-(\mu(\chi > x))$$

For the ask price one has to consider the integral

$$rvH_v = -\int_0^\infty x d\Gamma_+(\mu(\chi > x))$$


Bid and Ask as functions of Time for three different loss levels
800
700
600
500
400
300
200
100
0
0
50
100
150
Time in Years
Bid and Ask

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Unbounded liabilities, capital reserve requirements and the taxpayer put option. *Quantitative Finance* 12 (2012) 709–724

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