

# Derivatives, Diffusions, and Duality

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- While the results in this presentation are fairly new to me, I'm a bit concerned that they won't be new to all of you.
- I suspect that my results might be buried somewhere in the voluminous literature on backward SDE's.
- After you hear my talk today, I'll be grateful if someone can email me a particular page number where the results I'm presenting today are explained.

- Derivatives pricing was originated by Bachelier (1900) in a diffusion context.
- Still in a diffusion context, Merton (1973) showed that the market risk from selling a derivative can be eliminated by continuously revising a position in the derivative's underlying asset. In this talk, we refer to this continuously revised position as the derivative's delta.
- This talk focusses exclusively on European-style derivatives whose payoff is convex in the price of its underlying asset. We zero out carrying costs such as interest and dividends. In our univariate diffusion setting, the derivative's value function will also be convex.
- In this context, we give the sense in which a derivative's delta is dual to the price of its underlying asset.

- Suppose that an investor sells a univariate contingent claim with a convex payoff at the valuation date  $t = 0$ . The claim seller holds the short position in the claim statically until the claim matures at  $t = T$ .
- To eliminate the claim's risk in our univariate diffusion context, suppose that the claim seller also continuously trades in the spot market for the claim's underlying risky asset from  $t = 0$  to  $t = T$ .
- To finance purchases of the underlying risky asset, the claim seller borrows cash. These cash borrowings are offset by any sales of the underlying risky asset. The premium inflow from the initial sale of the contingent claim also offsets initial cash borrowings. In contrast, any final payout made by the claim seller adds to the final cash borrowings.
- In this talk, we give the sense in which the cash borrowings is dual to the value of the overlying contingent claim.

- We first introduce our two dualities at the function level, then graduate our pair of dualities to the level of stochastic processes.
- We use lower case letters to indicate (deterministic) functions and we use upper case letters to indicate (scalar) stochastic processes.
- For example,  $c(p, s)$  denotes the contingent claim value  $c$  as a function of the underlying's price  $p$  at time  $s$ . In contrast,  $C_t$  denotes the stochastic process for claim value, evaluated at time  $t$ . If  $P_t$  is the stochastic process describing the underlying's price, then  $C_t = c(P_t, t)$  for  $t \in [0, T]$ .

- For now, set  $s = 0$ , so  $c(p) \equiv c(p, 0)$  denotes the initial claim value  $c$  as a function of the initial price  $p$  of its underlying. Let  $q = c'(p)$  be the initial position in the underlying asset used to delta hedge the sale of the claim.
- Let  $\vartheta(q)$  be the initial cash borrowings, considered as a function of the initial position  $q$  in the underlying. The initial purchase of  $q$  units of the underlying adds to cash borrowings, while the initial sale of one unit of the claim subtracts, so:

$$\vartheta(q) = qp - c(p),$$

where  $q = c'(p)$  and hence  $p = c'^{-1}(q)$ .

- Since  $c$  is a convex function of  $p$ , we can interpret  $\vartheta(q)$  as the following Fenchel transform of  $c(p)$ :

$$\vartheta(q) = \sup_p [qp - c(p)].$$

- Recall that the cash borrowings function  $\mathfrak{c}(q)$  is the Fenchel transform of the claim value function  $c(p)$ :

$$\mathfrak{c}(q) = \sup_p [qp - c(p)] = qp - c(p),$$

where in the dual view, price  $p$  depends on position  $q$  since  $p = c'^{-1}(q)$ .

- By the envelope theorem,  $\mathfrak{c}'(q) = p$ . By the obvious convexity of  $\mathfrak{c}(q)$ ,  $\mathfrak{c}'(q)$  is increasing in  $q$ , and hence  $q = \mathfrak{c}'^{-1}(p)$ .
- Replacing  $q$  in the extremes of the top equation with  $\mathfrak{c}'^{-1}(p)$  and then solving for  $c(p)$ :

$$c(p) = p\mathfrak{c}'^{-1}(p) - \mathfrak{c}(\mathfrak{c}'^{-1}(p)) = \sup_q [qp - \mathfrak{c}(q)],$$

since  $\mathfrak{c}(q)$  is convex.



- Recall that the cash borrowings function  $\mathfrak{c}(q)$  and the claim value function  $c(p)$  can each be represented as:

$$\mathfrak{c}(q) = \sup_p [qp - c(p)] = qp - c(p), \text{ where price } p = c'^{-1}(q).$$

$$c(p) = \sup_q [qp - \mathfrak{c}(q)] = pq - \mathfrak{c}(q), \text{ where position } q = \mathfrak{c}'^{-1}(p).$$

- With price  $p$  and position  $q$  as conjugate variables, the convex cash borrowings function  $\mathfrak{c}(q)$  is dual to the convex claim value function  $c(p)$  and vice versa.

- As an example, the BMS call value function under zero carrying costs and strike  $K > 0$  is:

$$c(p) = pN(d_1(p)) - K(N(d_2(p))).$$

- The second equation and the envelope theorem imply that delta is simply:

$$c'(p) = N(d_1(p)),$$

and hence the cash borrowings function  $\mathfrak{c}(q) = KN(d_2(d_1^{-1}(N^{-1}(q))))$ .



- Assuming no frictions, no carrying costs, and no arbitrage, the FTAP says that there exists an equivalent martingale measure  $\mathbb{Q}$  such that all asset prices are martingales.
- In particular, we assume that the underlying asset price  $P$  is time-inhomogeneous univariate diffusion:

$$dP_t = a(P_t, t)dW_t, \quad t \geq 0,$$

where  $a(p, s) : \mathbb{R} \times \mathbb{R}^+ \mapsto \mathbb{R}$  is the dollar volatility function and  $W$  is a  $\mathbb{Q}$  standard Brownian motion.

- As a consequence, the claim value function  $c(p, s) \equiv E^{\mathbb{Q}}[f(P_T)|P_s = p]$  solves a terminal value problem and the claim value process  $C_t$  satisfies:

$$dC_t = c_1(P_t, t)dP_t = \mathfrak{Q}_t dP_t, \text{ where } \mathfrak{Q}_t = c_1(P_t, t) \text{ for } t \in [0, T].$$

- We say claim value  $C$  is a  $\mathbb{Q}$  martingale transform of price  $P$ .

- Recall that the claim value process  $C_t = E^{\mathbb{Q}}[f(P_T)|P_t]$  is a  $\mathbb{Q}$  martingale transform of its underlying asset price process  $P$ :

$$dC_t = \varphi_t dP_t, \text{ for } t \in [0, T],$$

where the position process  $\varphi_t$  is the claim's delta  $c_1(P_t, t)$ ,  $t \in [0, T]$ .

- In our complete market, the claim value  $C_t$  is the time  $t$  cost of its replicating portfolio:

$$C_t = \varphi_t P_t - \mathcal{D}_t, \text{ for } t \in [0, T],$$

where  $\mathcal{D}_t$  is the cash borrowing process. Integrating by parts:

$$dC_t = \varphi_t dP_t + P_t b \varphi_t - d\mathcal{D}_t \text{ for } t \in [0, T],$$

where  $b$  signifies the use of a backward Itô integral.

- The top equation implies:

$$d\mathcal{D}_t = P_t b \varphi_t, \text{ for } t \in [0, T].$$



- When the claim value process  $C_t = E^{\mathbb{Q}}[f(P_T)|P_t]$  is a  $\mathbb{Q}$  martingale transform of its underlying asset price process  $P$ :

$$dC_t = \varrho_t dP_t, \text{ for } t \in [0, T],$$

then the last slide showed that changes in cash borrowing arise purely from re-balancing the delta hedge:

$$d\mathcal{D}_t = P_t b \varrho_t, \text{ for } t \in [0, T].$$

- The backward increments  $b\varrho_t$  become forward increments if calendar time is run backwards, i.e. if  $\jmath = T - t$ , then:

$$d\hat{\mathcal{D}}_{\jmath} = \hat{P}_{\jmath} d\hat{\varrho}_{\jmath}, \text{ for } \jmath \in [0, T],$$

where hats denote reversed processes, i.e.  $\hat{\mathcal{D}}_{\jmath} = \mathcal{D}_t$ ,  $\hat{P}_{\jmath} = P_t$ ,  $\hat{\varrho}_{\jmath} = \varrho_t$ .

- If we regard  $\hat{\mathcal{D}}_{\jmath}$  as the dual of  $C_t$ ,  $\hat{P}_{\jmath}$  as the dual of  $P_t$ , and  $\hat{\varrho}_{\jmath}$  as the dual of  $\varrho_t$ , then this last equation dualizes the top one.
- Moreover, we will show that we can construct a reversed cash borrowing process  $\hat{\mathcal{D}}$  which is a  $\mathbb{Q}$  martingale transform of a reversed position process  $\hat{\varrho}$ !



- Again recall that the underlying asset price  $P$  is a time-inhomogeneous univariate diffusion under  $\mathbb{Q}$ :

$$dP_t = a(P_t, t)dW_t, \quad t \in [0, T].$$

- Assuming  $a(p, s)$  is differentiable in  $p$ , (heuristically) consider subtracting & adding  $a_1(P_t, t)dP_t dW_t$ :

$$dP_t = -a_1(P_t, t)dP_t dW_t + [a(P_t, t) + a_1(P_t, t)dP_t]dW_t, \quad t \in [0, T].$$

Substituting the top equation in the drift:

$$dP_t = -a_1(P_t, t)a(P_t, t)dt + a(P_t, t)bW_t, \quad t \in [0, T],$$

where recall the  $b$  indicates a backward Itô integral. When we reverse time as we do on the next slide, the backward Itô integral will become the usual forward Itô integral with respect to a standard Brownian motion.

- Recall the SDE for the price  $P$  of the underlying asset:

$$dP_t = -a_1(P_t, t)a(P_t, t)dt + a(P_t, t)bW_t, \quad t \in [0, T],$$

where  $a(p, s) : \mathbb{R} \times [0, T] \mapsto \mathbb{R}$  is the function relating the asset's dollar volatility to the price level  $p$  and to the clock  $s$  which indexes the underlying asset's price process. The coefficient  $a(P_t, t)$  multiplying backward increments  $bW_t$  is a stochastic process  $A_t$ .

- Now for  $t \in [0, T]$ , let  $\tau = T - t$  be the time to maturity (aka term) of the claim. Let  $\hat{P}_\tau = P_t$  be the reversed price process and let  $\hat{A}_\tau = A_t$  be the reversed vol process, both for  $\tau \in [0, T]$ . We define this pair jointly by thinking of  $\hat{A}_\tau = a(\hat{P}_\tau, \tau)$  as opposed to  $\hat{A}_\tau = a(\hat{P}_\tau, T - \tau)$ .
- With this convention in mind, existence results on backward SDE's suggest there exists a standard Brownian motion  $\hat{B}$  in the term clock  $\tau$  such that:

$$d\hat{P}_\tau = a_1(\hat{P}_\tau, \tau)a(\hat{P}_\tau, \tau)d\tau + a(\hat{P}_\tau, \tau)d\hat{B}_\tau, \quad \tau \in [0, T].$$



- Recall our assumption that the underlying asset price  $P$  is time-inhomogeneous univariate diffusion under  $\mathbb{Q}$ :

$$dP_t = a(P_t, t)dW_t, \quad t \geq 0.$$

- The claim value function  $c(p, s) \equiv E^{\mathbb{Q}}[f(P_T)|P_s = p]$  will be  $C^{2,1}$ . The claim value process  $C_t = c(P_t, t)$  for  $t \in [0, T]$ , so by Itô's formula:

$$dC_t = \left[ \frac{a^2(P_t, t)}{2} c_{11}(P_t, t) + c_2(P_t, t) \right] dt + c_1(P_t, t)a(P_t, t)dW_t, \quad t \in [0, T].$$

- By FTAP,  $C$  is a  $\mathbb{Q}$  martingale, so the claim value function  $c(p, s)$  solves the following well-known backward linear second-order parabolic PDE:

$$\frac{a^2(p, s)}{2} c_{11}(p, s) + c_2(p, s) = 0, \quad p \in \mathbb{R}, s \in [0, T].$$

- Barring pathologies on the dollar volatility function  $a(p, s)$  and the terminal payoff function  $f(p)$ , the claim value function  $c(p, s)$  is determined by the PDE and the terminal condition  $c(P, T) = f(p)$ . One can numerically solve the terminal value problem by recursing backward in calendar time.



- Recall that the claim value function  $c(p, s)$  solves the following PDE:

$$\frac{a^2(p, s)}{2} c_{11}(p, s) + c_2(p, s) = 0, \quad p \in \mathbb{R}, s \in [0, T],$$

subject to  $c(p, T) = f(p), p \in \mathbb{R}$ .

- Recalling that the position function  $q(p, s) = c_1(p, s)$ , differentiating w.r.t.  $p$  implies that:

$$\frac{a^2(p, s)}{2} q_{11}(p, s) + a(p, s) a_1(p, s) q_1(p, s) + q_2(p, s) = 0, \quad p \in \mathbb{R}, s \in [0, T],$$

subject to  $q(p, T) = f'(p), p \in \mathbb{R}$ .

- Now recall the Itô SDE for the reversed price process:

$$d\hat{P}_t = a_1(\hat{P}_t, t) a(\hat{P}_t, t) dt + a(\hat{P}_t, t) d\hat{B}_t, \quad t \in [0, T].$$

- Letting  $\hat{Q}_t = q(\hat{P}_t, t)$  for  $t \in [0, T]$ , the above PDE implies that  $\hat{Q}_t$  is a  $\mathbb{Q}$  local martingale in the  $t$  clock.





- Recall that the reversed price process is defined so that its Itô SDE is:

$$d\hat{P}_t = a_1(\hat{P}_t, t)a(\hat{P}_t, t)dt + a(\hat{P}_t, t)d\hat{B}_t, \quad t \in [0, T],$$

and that the reversed position process  $\hat{q}_t = c_1(\hat{P}_t, t)$  is a  $\mathbb{Q}$  local martingale in the  $t$  clock. As a result, Itô's formula implies:

$$d\hat{q}_t = q_1(\hat{P}_t, t)a(\hat{P}_t, t)d\hat{B}_t, \quad t \in [0, T].$$

- The assumed convexity of the claim value function  $c(p, s)$  in price  $p$  implies that the position function  $q = c_1(p, s)$  is increasing in price  $p$  and hence is invertible:  $p = c_1^{-1}(q, s)$ . Since  $\hat{q}_t = c_1(\hat{P}_t, t)$ ,  $\hat{P}_t = c_1^{-1}(\hat{q}_t, t)$ , hence:

$$d\hat{P}_t = q_1(c_1^{-1}(\hat{q}_t, t), t)a(c_1^{-1}(\hat{q}_t, t), t)d\hat{B}_t = \epsilon(\hat{q}_t, t)d\hat{B}_t, \quad t \in [0, T],$$

where  $\epsilon(q, s) = q_1(c_1^{-1}(q, s), s)a(c_1^{-1}(q, s), s)$  is the function relating the normal volatility of position to position  $q$  and clock  $s$ .

- Since the volatility of the reversed position process  $\hat{q}$  has been expressed as a function of just the reversed position  $\hat{q}_t$  and reverse time  $t$ , the reversed position process  $\hat{q}$  is a driftless diffusion under  $\mathbb{Q}$ .



- Recall that with calendar time  $t$  running forward, all changes in the cash borrowings process  $\mathcal{D}$  arise solely due to changes in position  $\mathcal{Q}$ :

$$d\mathcal{D}_t = P_t b \mathcal{Q}_t, \quad t \in [0, T].$$

- Replacing calendar time  $t$  with term  $\hat{t} \equiv T - t$ , price  $P_t$  with reversed price  $\hat{P}_{\hat{t}}$ , and position  $\mathcal{Q}_t$  with reversed position  $\hat{\mathcal{Q}}_{\hat{t}} = c_1(\hat{P}_{\hat{t}}, \hat{t})$ , we get an Itô SDE which determines the reversed cash borrowing process  $\hat{\mathcal{D}}_{\hat{t}} = \mathcal{D}_t$ :

$$d\hat{\mathcal{D}}_{\hat{t}} = \hat{P}_{\hat{t}} d\hat{\mathcal{Q}}_{\hat{t}}, \quad \hat{t} \in [0, T].$$

- Since the reversed position process  $\hat{\mathcal{Q}}$  is driftless under  $\mathbb{Q}$  and in clock  $\hat{t}$ , it follows that the reversed cash borrowing process  $\hat{\mathcal{D}}_{\hat{t}} = \hat{\mathcal{Q}}_{\hat{t}} \hat{P}_{\hat{t}} - \hat{\mathcal{C}}_{\hat{t}}$  is also driftless. Furthermore, differentiating the cash borrowing function  $\mathfrak{c}(q) = c_1^{-1}(q)q - c(c_1^{-1}(q))$  with respect to position  $q$  returns price as a function of position  $\mathfrak{c}'(q) = c_1^{-1}(q)$ . As a result, the Itô SDE can be written as:

$$d\hat{\mathcal{D}}_{\hat{t}} = \mathfrak{c}_1(\hat{\mathcal{Q}}_{\hat{t}}, \hat{t}) d\hat{\mathcal{Q}}_{\hat{t}}, \quad \hat{t} \in [0, T].$$



- Supposing under  $\mathbb{Q}$  that the underlying price  $P$  is driftless diffusion:

$$dP_t = a(P_t, t)dW_t, \quad t \in [0, T],$$

then changes in claim value  $C_t = c(P_t, t)$  arise purely from gains:

$$dC_t = c_1(P_t, t)dP_t, \quad t \in [0, T].$$

- The price process  $P$  & the claim value  $C$  are both  $\mathbb{Q}$  local martingales. In contrast, the position process  $q_t = c_1(P_t, t)$  & the cash borrowings process  $\mathcal{D}_t = q_t P_t - C_t$  can both drift.

- However when time reverses into term  $\tau = T - t$ , there exists a reversed price process  $\hat{P}_\tau$  which drifts so that the reversed position process  $\hat{q}_\tau = c_1(\hat{P}_\tau, \tau)$  is driftless diffusion:

$$d\hat{q}_\tau = \epsilon(\hat{q}_\tau, \tau)d\hat{B}_\tau, \text{ where } \epsilon(q, s) = q_1(c_1^{-1}(q, s), s)a(c_1^{-1}(q, s), s), \tau \in [0, T].$$

- Furthermore, the reversed claim value  $\hat{C}_\tau = c(\hat{P}_\tau, \tau)$  drifts so that the reversed cash borrowings process  $\hat{\mathcal{D}}_\tau = \hat{q}_\tau \hat{P}_\tau - \hat{C}_\tau$  is a  $\mathbb{Q}$  local martingale:

$$d\hat{\mathcal{D}}_\tau = \varrho(\hat{q}_\tau, \tau)d\hat{q}_\tau, \text{ where } \varrho(q, \tau) = c_1^{-1}(q, s)q - c(c_1^{-1}(q, s), s), \tau \in [0, T].$$



- Recall that the claim value function  $c(p, s)$  solves the following PDE:

$$\frac{a^2(p, s)}{2} c_{11}(p, s) + c_2(p, s) = 0, \quad p \in \mathbb{R}, s \in [0, T].$$

- This PDE is numerically solved by running  $s$  backwards from the terminal condition  $c(p, T) = f(p), p \in \mathbb{R}$  to the initial value  $c(p, 0), p \in \mathbb{R}$ .
- Suppose one changes all 3 variables in the above PDE defining  $z = T - s$ ,  $q = c_1(p, s)$ , and  $\varpi(q, z) = qp - c(p, s)$ . Then one can derive the PDE:

$$\frac{\epsilon^2(q, z)}{2} \varpi_{11}(q, z) + \varpi_2(q, z) = 0, \quad \text{where } \epsilon(q, z) = q_1(c_1^{-1}(q, s), z) a(c_1^{-1}(q, s), s),$$

for  $q \in \mathbb{R}, z = T - s$ , and  $s \in [0, T]$ , which confirms that the reversed cash borrowing process  $\hat{\mathcal{C}}_t = \hat{q}_t \hat{P}_t - \hat{C}_t, t \in [0, T]$  is driftless under  $\mathbb{Q}$ .

- Recall that when cash borrowing is expressed as a function  $\vartheta$  of position  $q$  and term  $z$ , it solves the following second order linear parabolic PDE:

$$\frac{\sigma^2(q, z)}{2} \vartheta_{11}(q, z) + \vartheta_2(q, z) = 0, \quad q \in \mathbb{R}, z \in [0, T].$$

- This PDE can be numerically solved by running term  $z$  backwards from the terminal condition  $\vartheta(q, T) = q c_1^{-1}(q, 0) - c(c_1^{-1}(q, 0), 0)$ ,  $q \in \mathbb{R}$ .
- So if one wanted to update the model value of a claim from yesterday to today, one need only transform yesterday's claim value profile to yesterday's cash borrowings profile, shrink term by one day via the above PDE, and then transform back to today's claim value profile. This is usually much more computationally efficient than directly re-solving for claim value by running backwards in calendar time from the maturity date.

- In the standard (primal) approach to contingent claim valuation, the underlying price process  $P$  is specified as some driftless diffusion.
- One then solves a backward PDE for claim value as a function of the price of the underlying and calendar time.
- If one wishes, the solution to this PDE can be differentiated to obtain claim delta as a function of price and time.
- One can obtain cash borrowings by price and time by subtracting the claim value from the product of delta and price.
- If the claim value function is convex, one can numerically invert the increasing function relating delta to the price of the underlying to obtain the price of the underlying as a function of delta and time.
- As a result, one can also obtain cash borrowings as a function of delta and time.

- In the dual approach to contingent claim valuation, the reversed position process  $\hat{Q}$  is directly specified as some driftless diffusion.
- One then solves a backward PDE for cash borrowings as a function of the position and term.
- The solution to this PDE can be differentiated to obtain price of the underlying as a function of position and term.
- One can obtain claim value by delta and term by subtracting the cash borrowings from the product of position and the price of the underlying.
- If the claim value function is convex, then one can numerically invert the increasing function relating price of the underlying to delta in order to obtain delta as a function of the underlying price and time.
- As a result, one can also obtain the claim value as a function of the underlying price and time.

- Some technical issues arise when one wishes to specifically apply the dual approach to European call valuation.
- The payoff  $f(p) = (p - K)^+$  is convex in  $p$ , but it is not strictly convex. As a result, the reversed call delta process need not be driftless at zero term (=expiry), but reversed call delta is a driftless diffusion whenever term is strictly positive.
- In diffusion models for price, the reversed call delta process is a term-inhomogeneous diffusion whose state space is the unit interval  $[0, 1]$ . Furthermore, this diffusion is driftless after term zero. It is not immediately obvious how to pick such a process, but fortunately, this state space is exactly the state space in which probabilities reside. One can begin with a time homogeneous diffusion on any regular domain, calculate analytically friendly objects like hitting probabilities over infinite horizons, and then finally deterministically change the clock that these probabilities run on to induce a term-inhomogeneous driftless diffusion on the state space  $[0, 1]$  and over a finite time interval  $(0, T]$ .



- We showed that in univariate driftless diffusion models for an underlying asset price, the claim's delta can be regarded as a conjugate process of price, particularly when this position process is run backwards in time.
- Similarly, the cash borrowing process can be regarded as a dual process for claim value, again when time is reversed.
- This alternative view has advantages in understanding greeks, numerically updating claim values, and perhaps in developing new analytically tractable models.
- Just as Lévy processes are more easily analyzed in Fourier space, it would be interesting to explore which processes are best understood in the Fenchel-Legendre space that we have just outlined.
- Furthermore, trading strategies which are purely delta-based or contingent claims whose payoffs depend on delta (eg. American options) are perhaps more easily analyzed in our dual space.