

Joint Risk Neutral Laws and Hedging

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The Hedging Problem

- Much has been written on hedging both when it is possible and when it is not.
- However, less attention is given to constructing useful hedges for prespecified or existing exposures in high dimensions using options as a static hedge.
- Carr and Wu (2004) have considered hedging long dated options with positions in shorter term options, working in one dimension.
- In this context we have completeness as arbitrary functions of the underlying are spanned by the options (Green and Jarrow (1987), Nachman (1988)).

Multidimensional Hedging

- To focus attention we will consider a target exposure written on 9 underliers representing the level of sector *ETF's* three months hence.
- The exposure will be to a positive random variable whose logarithm is a quadratic function of the 9 logarithmic returns.
- The associated matrix of second order cross partials will be chosen to display both convex and concave directions by ensuring the presence of both positive and negative eigenvalues. The hedging assets will be the option surfaces on all 9 underliers.
- In this multidimensional context options on the underliers do not span the target cash flow. We thereby have a context of incompleteness as studied in Carr, Geman and Madan (2001), or Jaschke and Kuchler (2001).

The Hedging Probability Measure

- The first question to be faced is the choice of the probability law under which we study the hedging problem.
- Numerous authors have employed the physical probability to manage the risk or approximate the target cash flow (Bertsimas, Kogan and Lo (1999), Biagini, Guasoni and Pratelli (2000), Föllmer and Sondermann (1986)).
- We question the choice of the physical measure and argue that it is possibly not the recommended choice.
- Under the physical probability most hedging assets have a nonzero expectation that is either positive or negative.

- As a consequence algorithms seeking to hedge pre-specified exposures can be induced into investing in or shorting the hedging assets with a view to generating returns, and setting aside the hedging and risk management concerns.
- We argue that it is critical to proceed under a risk neutral measure as a base measure under which to study the hedging problem.

The Hedging Criterion

- The second question to be considered is the choice of a criterion for the hedge. Many criteria have been proposed and studied.
- These criteria trade risks and returns in a variety of ways and generally attempt to model the risk attitudes and preferences of the party holding the residual risk.
- Examples include the use of mean variance preferences (Heath, Platen and Schweizer (1999)), others measure risk by the conditional value at risk (Rockafellar and Uryasev (2000)), while some writers employ expected utility in the presence or absence of background or endowment risk (Schachermayer (2002)).

- These preference based criteria combine scale and direction issues possibly altering the hedge if the position is scaled.
- We shall, however, here take a more market oriented approach that presumes that one may trade with the market at a wide range of scales without an adjustment to the terms of trade.

Market Criterion

- Unlike the classical model of the market where one trades in both directions at the same price we suppose the terms of trade depend on the direction of trade but not its size.
- In this regard we follow Cherny and Madan (2009, 2010) and model the set of marketed cash flows as a convex cone containing the nonnegative cash flows.
- Such a model for the set of marketed cash flow results in closed forms for the bid and ask prices of cash flows and we may then either maximize the bid price for the hedge or minimize the ask price for the hedge.
- Since we see our target cash flow as a potential liability we minimize its ask price here.

Risk Neutral Multidimensional Laws

- The third question that arises is that of modeling risk neutrally the joint law of nine underliers.
- As we shall take positions in options on the underliers to hedge the risk exposure we seek a joint law that will calibrate simultaneously all option surfaces on a single day.
- We have employed many models in the exponential affine class to calibrate the surface of options on a single underlier. Here we develop multivariate exponential affine models. Such exercises require one to model dependence across a set of underliers.
- We note, however, that options only trade on each underlier separately and there isn't a liquid market for options on portfolios that may explicitly price correlations.

- Hence, the approach we take is a mixture of time series and option data analysis.

Risk Neutral Multidimensional Laws cont'd

- Time series data are employed to model the martingale component of the log price relative as a linear mixture of independent variates.
- The independent components are identified by an application of independent components analysis. Such an analysis delivers as an output the matrix of asset exposures to component risks.
- For our risk neutral law we model the independent components using a Sato process (Carr, Geman, Madan and Yor (2007)) taking the exposure matrix as estimated from the time series analysis.

- Thus we develop a joint characteristic function for all underliers allowing one to compute option prices for any strike, any maturity and any underlier.
- We then calibrate this joint law to all surfaces simultaneously. Our illustration employs 9 underliers and a Sato process based on the variance gamma law at unit time for each component.
- The Sato process uses a single scaling parameter for all underliers and we have a model with 28 parameters for the nine surfaces.

Joint Risk Neutral Laws

- Our starting point is the recognition that in the Gaussian or Brownian case, one builds dependence by modeling multidimensional risks as a linear combination of independent Gaussian variates.
- Here we generalize to considering a linear combination of independent processes with independent increments.
- When these independent processes are in addition time homogeneous, they are Lévy processes.
- However, given that we wish to model risk neutral option surfaces across strike and maturity we consider the particular class of time inhomogeneous processes known as Sato processes (Carr, Geman, Madan and Yor (2007)).

- We shall also consider Lévy processes that are particularly suitable when the focus is on a single maturity.

ICA

- In identifying the joint risk neutral law one has to determine both the coefficients of the linear combination and the risk neutral law of the supposed independent components.
- In general this gives us too many parameters to be inferred from the option surfaces. We shall instead determine the matrix of asset exposures to independent risks from a time series analysis and then we employ this matrix as an input in determining the risk neutral law for the components.
- For the time series analysis we follow Madan and Yen (2008) and Madan (2006) in applying independent components analysis to identify both the exposure matrix and the data on the independent components.

ICA Remarks

- Independent components analysis (ICA, Hyvärinen, Karhunen and Oja (2001)) and in particular the fast ICA algorithm (Hyvärinen (1999)) exploits the observation that signals have non-Gaussian distributions that approach Gaussian distributions when contaminated by noise.
- Hence the algorithm first performs a principal components analysis (PCA), and noting that every rotation of a PCA is an equivalent PCA, the ICA procedure seeks a rotation matrix that maximizes a metric of non-Gaussianity to try and get back to the original signal.
- The procedure has been tested in the signal processing context and it has been demonstrated that unlike PCA, ICA can recover original signals.

- It is also noted that if the original data is multivariate Gaussian, then there are no signals to be found and the algorithm fails.
- Fortunately, for financial data on daily returns the Gaussian assumption fails and an ICA delivers the required exposure matrix.

Models for the joint risk neutral laws of n assets

- Let $S(t) = (S_1(t), \dots, S_n(t))$ denote the vector of n stock prices at time t , with $X_k(t) = \ln S_k(t)$ for $k = 1, \dots, n$.
- We suppose that the vector $X(t) = (X_1(t), \dots, X_n(t))$ is a linear combination of independent processes $Y_k(t)$ and

$$X(t) = AY(t).$$

- One may employ time series analysis to identify the exposure matrix A and the statistical or physical law of the process $Y(t)$.
- Since our interest is in the risk neutral law we employ time series data just to identify the exposure matrix A .

The first model (Lévy) termed VGICA

- A first risk neutral law for $Y(t)$ models each component as an independent Lévy process and here we employ the variance gamma process.
- We then have $3n$ parameters $\sigma_k, \nu_k, \theta_k$ such that the zero mean process for $Y_k(t)$ is

$$Y_k(t) = \theta_k (g_k(t) - t) + \sigma_k W_k(g_k(t))$$

where $g_k(t)$ are independent gamma processes with unit mean rate and variance rate ν_k and $W_k(t)$ are independent Brownian motions also independent of the gamma processes.

- The characteristic functions for the processes $Y_k(t)$

are

$$\begin{aligned}\phi_{Y_k}(u_k, t) &= E[\exp(u_k Y_k(t))] \\ &= \exp(t\psi_k(u_k)) \\ \psi_k(u_k) &= \frac{1}{\nu_k} \ln \left(1 - iu_k\theta_k - \frac{\sigma_k^2 \nu_k}{2} \right) - \theta_k.\end{aligned}$$

- From the independence of the processes Y_k one easily derives the joint characteristic function for the vector process $X(t)$ as

$$\begin{aligned}E[\exp(iu'X(t))] &= \prod_{k=1}^n \exp \left(t\psi_k \left(\sum_{j=1}^n u_j A_{jk} \right) \right) \\ &= \text{def} \phi_X(u, t)\end{aligned}$$

- The stock prices for the n assets are related to the processes $X(t)$ by

$$S_j(t) = S_j(0) \exp \left((r - q_j + \omega_j)t + X_j(t) \right)$$

where the convexity correction coefficients are

$$\omega_j = - \sum_{k=1}^N \psi_k(-iA_{jk}).$$

- The joint characteristic function for the logarithm of all the stocks under the *VGICA* model is then

$$\begin{aligned} & E \left[\exp \left(iu' \ln(S(t)) \right) \right] \\ &= \exp \left(i \sum_{j=1}^n u_j \left(\ln S_j(0) + (r - q_j + \omega_j)t \right) \right) \\ & \quad \times \phi_X(u, t). \end{aligned}$$

The second model (Sato) VGSSDICA

- We now consider a Sato process for the independent components.
- In this case we model the centered law for the logarithm of the stock as before but the components of the vector $Y(t)$ are now not Lévy processes but Sato processes.
- We therefore obtain the marginal distribution of each component at each time t by scaling and suppose that

$$Y(t) \stackrel{(d)}{=} t^\gamma Y(1)$$

where the scaling coefficient is common across components.

- In this case we have scaling as a vector for now

$$\begin{aligned} X(t) &= AY(t) \\ &\stackrel{(d)}{=} At^\gamma Y(\mathbf{1}) \\ &= t^\gamma AY(\mathbf{1}) \\ &= t^\gamma X(\mathbf{1}) \end{aligned}$$

The Multivariate Lévy System

- Each component is now a process of independent but time inhomogeneous increments with a jump compensating Lévy system for the k^{th} component Y_{kt} of the form

$$L_k(y_k, t)dy_k dt.$$

- The specific form of the function L_k may be obtained from Carr, Geman, Madan and Yor (2007) in terms of the Lévy density $l_k(y)$ of the unit time random variable $Y_k(1)$ as

$$L_k(y_k, t) = -\text{sign}(y_k) \frac{h'_k\left(\frac{y_k}{t^\gamma}\right) \gamma}{t^{1+\gamma}}, \quad y_k \neq 0,$$
$$h_k(y_k) = |y_k| l_k(y_k).$$

- By independence the Lévy system for the vector process Y_t is given by

$$L_Y(y, t)dydt = dt \sum_{k=1}^n L_k(y_k, t)dy_k$$

- We may write the Lévy system for $X(t)$ an additive process as

$$L_X(x, t)dxdt = dt dx \frac{1}{\det(A)} \sum_{k=1}^n L_k \left((A^{-1}x)_k, t \right).$$

The Characteristic Function

- The joint characteristic function of the vector X_t is now given by

$$\begin{aligned} E \left[e^{iu'X(t)} \right] &= \prod_{k=1}^n \exp \left(\sum_{j=1}^n \psi_k(t^\gamma u_j A_{jk}) \right) \\ &= \text{def} \tilde{\phi}_X(u, t) \end{aligned}$$

and the convexity correction term is

$$\omega_j(t) = - \sum_{k=1}^N \psi_k(-it^\gamma A_{jk})$$

- The joint characteristic function for the logarithm of

the vector $\ln(S(t))$ under $VGSSDICA$ is

$$\begin{aligned} & E \left[\exp \left(iu' \ln(S(t)) \right) \right] \\ = & \exp \left(i \sum_{j=1}^n u_j \left(\ln S_j(0) + ((r - q_j)t + \omega_j(t)) \right) \right) \\ & \times \tilde{\phi}_X(u, t). \end{aligned}$$

Estimation Procedure

- The risk neutral estimation of *VGICA* or *VGSSDICA* begins with a time series analysis conducted for the purpose of estimating the exposure matrix A . For this we take times series data on daily returns

$$\begin{aligned}x_t &= (x_{1t}, \dots, x_{nt}) \\ &= \left(\ln \left(\frac{S_1(t)}{S_1(t-1)} \right), \dots, \ln \left(\frac{S_n(t)}{S_n(t-1)} \right) \right)\end{aligned}$$

for T days.

- We form the matrix of demeaned daily return data by subtracting the sample means for each component to construct the $T \times n$ matrix R of centered returns. We apply to this matrix the fast ICA algorithm that returns a matrix A and data on the independent components constructed as

$$y_t = A^{-1}x_t.$$

- The matrix A composes a PCA matrix with a search for an optimal rotation matrix that successively maximizes the expectation of the logarithm of the cosine of the implied factor return y_{kt} in the sample.

- With the exposure matrix estimated from such a fast ICA algorithm we may now employ the Fourier transform methods of Carr and Madan (1999) to price options on all strikes, maturities and underliers to simultaneously calibrate all option surfaces to identify the joint risk neutral law.
- The joint characteristic functions and take as inputs respectively 27 and 28 parameters for the 9 sets of VG parameters with 9 underliers and for the Sato process the scaling coefficient γ .
- When computing an option price, in addition to the strike and maturity one must indicate as an input the stock or underlying asset to enable the pricing routine to construct the appropriate marginal characteristic function that is inverted for the option price.
- The risk neutral parameters for the joint law are estimated by minimizing the root mean square error of market price from model price for all option surfaces taken together.

The Hedging Problem

- This section describes in detail the specific hedging problem to be solved.
- We present in turn the target cash flow to be hedged, the structure of the hedging assets, and the hedge objective.

The Target Cash Flow

- The target function to be hedged on the joint space of all underliers is typically some estimated value function some time in the future.
- We illustrate here by taking a general translog function on the underliers defined as

$$\ln(f(S)) = \sum_{i=1}^n \alpha_i \ln \left(\frac{S_i}{S_{i0}} \right) + \frac{1}{2} \sum_{ij} \beta_{ij} \ln \left(\frac{S_i}{S_{i0}} \right) \ln \left(\frac{S_j}{S_{j0}} \right).$$

We wish to hedge the multivariate function $f(S)$ seen as a function of the stock prices of the underliers some three months out.

The Hedging Assets

- We take for hedging assets the n underliers and options on these underliers with a three month maturity and a range strikes. We may standardize the problem by taking all initial prices at unity.
- The options then have strikes expressed in moneyness with strikes below unity being put options and above unity we employ call options. We treat the underlying model as that for forward prices and work with zero rates and dividend yields.
- We take the strikes to range from .75 to 1.25 in steps of 0.025. We then have 21 options on each of 9 underliers with 189 options and 198 hedging assets for our 9 underliers.
- Viewing the underlier as a zero strike call we have 22 strikes per underlier.

- The terminal option payoff to strike k_j for underlier i is

$$c_{ij}(S_i) = (S_i - k_j)^+ + \mathbf{1}_{0 < k_j < 1} (k_j - S_i)$$

and for hedge positions θ_{ij} in the option with strike k_j on the i^{th} underlier the hedge payoff is

$$g(S, \theta) = \sum_i \sum_j \theta_{ij} (c_{ij}(S_i) - w_{ij})$$

where w_{ij} is the market price of option j on the i^{th} underlier.

- Under the models we shall consider the target cash flow is not in the span of the possible hedge cash flows. For every possible candidate hedge position $\theta = (\theta_{ij})$ the residual cash flow

$$r(S, \theta) = g(S, \theta) - f(S)$$

is nonzero.

The Hedge Objective

- We need to formulate an objective function to evaluate different possibilities for the residual cash flows.
- In complete markets the residual net of hedge costs is zero and we just find the replicating portfolio and value the cash flow $f(S)$ at the cost of replication $\sum_i \sum_j w_{ij}$. The hedge is clear.
- For the current context of incomplete markets this is not possible and the hedge is not clear.

Hedge Objective

- Many suggestions for a hedging criterion have been made in the literature, including minimizing variance, super or sub replicating, minimizing some other norm, or maximizing expected utility.
- The objectives of super or sub replicating are often unrealistic, in that they may not even be possible.
- The various norms treat overshoots and shortfalls symmetrically though only shortfalls are a problem.
- Expected utility as a criterion we find unsatisfactory. It is focused on modeling the preferences of a particular person.
- Furthermore the hedge, its cost and associated reservation prices for risk vary with the size of the trade.

Modeling Markets

- We follow instead a market oriented approach that shifts attention from modeling risks to be held by individuals to risks held in markets on terms defined by markets.
- The essentials of this theory are set out in Cherny and Madan (2010) following on from Carr, Geman and Madan (2001) and Cherny and Madan (2009).
- The main idea of this approach to modeling markets begins on replacing the law of one price for liquid markets by the law of two prices.
- Hence market participants may still trade with the market any amount at the going price with the proviso that the price now depends on the direction of trade.

- One then observes that the set of cash flows seen as random variables that one may deliver to market at zero cost form a convex cone of random variables that contains the set of nonnegative cash flows.

Bid and Ask Prices

- Formally the cash flows acceptable to the market at zero cost are random variables X with the property that $E^Q[X] \geq 0$ for all measures Q in a convex set of measures \mathcal{D} that support the acceptability.
- Given a class of \mathcal{H} of hedge cash flows attainable by trading in markets at zero cost Cherny and Madan (2010) define the risk neutral measures \mathcal{R} as all measures satisfying $E^Q[H] = 0$ for all $H \in \mathcal{H}$.
- They then show that the ask price for X , $a(X)$ is given by

$$a(X) = \sup_{Q \in \mathcal{D} \cap \mathcal{R}} E^Q[X].$$

- Similarly the bid price is given by

$$b(X) = \inf_{Q \in \mathcal{D} \cap \mathcal{R}} E^Q[X].$$

- The risk neutral measures play an important role in the formulation of these pricing problems and it is critical that \mathcal{D} is large enough to in fact meet the set of risk neutral measures.
- In fact if $\mathcal{D} \cap \mathcal{R}$ is empty there is a hedging asset that is strictly acceptable to the market and this should not be the case if the hedging assets are correctly priced.

Abstract Hedging Problem

- One may formulate the hedging problem as that of minimizing the post hedge ask price whereby

$$a(X) = \inf \left\{ a \mid E^Q[a + H - X] \geq 0, \text{ all } Q \in \mathcal{D} \right\}.$$

- The corresponding hedging problem associated with the bid price is

$$b(X) = \sup \left\{ b \mid E^Q[X - b - H] \geq 0, \text{ all } Q \in \mathcal{D} \right\}.$$

Operational Hedging Problem

- To make these market oriented hedging problems for the bid and ask price operational one has to define operational cones of cash flows acceptable to the market. For this we begin with a single risk neutral measure Q_0 that one identifies and we force a non-empty intersection of \mathcal{D} with \mathcal{R} by placing this base measure in our set \mathcal{D} .
- Now accepting all cash flows with a positive expectation under this single measure is too wide a cone and it is in fact a half space. We take the set of zero cost marketable cash flows to be smaller by requiring a positive expectation under a much larger class of measures.
- Following Cherny and Madan (2009,2010) we define acceptable cash flows in terms of the distribution function $F_X(x)$ of our cash flow X under Q_0 .

- For a specific concave distribution function $\Psi(u)$ defined on the unit interval and mapping to the unit interval we define X to be acceptable just if

$$\int_{-\infty}^{\infty} x d\Psi(F(x)) \geq 0,$$

or the expectation of the cash flow is positive under a concave distortion of its distribution function.

- When working with distribution functions one may identify the base measure to be the uniform distribution on the unit interval and identify distribution functions $F(x)$ with their inverses $G(u) = F^{-1}(u)$. One then speaks equivalently of acceptable inverse distribution functions. These are now such that we have a positive expectation for all measures with densities $Z(u)$ where the associated distribution function $L' = Z$ satisfies $L \leq \Psi$ Cherny and Madan (2010).

The Distortion Used

- The particular concave distortion studied in Cherny and Madan (2009, 2010) that we employ here is termed *minmaxvar* and is defined by

$$\Psi^\gamma(u) = 1 - (1 - u^{\frac{1}{1+\gamma}})^{1+\gamma}.$$

- In terms of this distortion one may define the ask price problem as

$$\begin{aligned} a(X) &= \min_{\theta_{ij}} - \int_{-\infty}^{\infty} x d\Psi^\gamma(F_X(x)) \\ X &= r(S, \theta). \end{aligned}$$

- We solve the ask price hedge problem by numerical optimization to determine the hedge positions θ_{ij} in the 198 hedging assets.

Calibration of VGICA and VGSSDICA

- Time series data on the daily levels of sector ETF's for a thousand days from 2005 January 3 to 2009 June 5 were employed to construct an ICA decomposition of returns taken as log price relatives. The ETF's employed were xlb, xle, xlf, xli, xlk, xlp, xlu, xlv, and xly.
- We present in Table 1 by way of a summary the levels of annualized mean, volatility, skewness, and kurtosis for the 9 ETF's in alphabetical order and the 9 ICA factors in the order of extraction.

TABLE 1

Summary Statistics returns and ICA factors

	Mean		Volatility		Skewness		Kurtosis	
xlb	-0.01	-0.50	0.32	18.53	-0.19	0.90	9.71	29.91
xle	0.03	7.32	0.39	16.43	-0.53	-0.53	10.30	11.98
xlfi	-0.23	-2.24	0.49	16.63	0.07	-0.20	10.49	8.67
xli	-0.06	4.04	0.27	19.86	-0.22	-0.13	9.30	6.84
xlk	-0.03	2.97	0.26	26.63	0.06	-0.04	10.43	8.29
xlp	-0.02	1.09	0.14	15.58	-0.42	0.13	22.59	6.83
xlu	-0.04	-0.08	0.24	17.01	0.50	-0.13	13.37	4.45
xlvi	-0.05	-7.28	0.20	20.32	-0.09	-0.11	19.32	5.09
xly	-0.09	-2.45	0.28	15.71	-0.26	0.12	8.89	5.05

- We see from Table 1 that the initial factors that are the most non-Gaussian have substantial levels of skewness and kurtosis.
- Table 2 presents the matrix A of factor exposures needed for the calibration of the risk neutral joint law under the $VGICA$ and $VGSSDICA$ models.
- We also estimated by maximum likelihood marginal variance gamma laws on the time series of the independent components as given by equation. We thus have a full specification of the physical law for the martingale component of log price relatives as a linear mixture on independent variance gamma processes.

TABLE 2

Factor Exposure Matrix extracted from an application of Independent Components Analysis to daily returns in basis points

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	Factor 7	Factor 8	Factor 9
xlb	111.67	-84.13	-27.88	73.56	-22.14	-52.94	87.18	-54.41	0.11
xle	149.75	-95.98	-89.79	41.53	-31.07	-40.51	18.50	-78.43	-105.58
xlf	88.06	-279.65	-43.45	-5.26	-24.93	-57.04	38.20	-1.12	42.49
xli	93.27	-79.23	-32.70	22.70	-20.49	-99.67	18.24	-15.19	23.81
xlk	87.21	-79.25	-74.34	53.14	-12.85	-29.66	-3.51	-18.31	49.41
xlp	52.26	-38.34	-53.90	-21.12	24.58	-28.64	29.77	-18.15	27.18
xlu	105.00	-39.35	-69.66	9.99	3.24	-14.49	42.42	45.74	-21.72
xlv	105.08	-38.08	-11.23	-26.78	-19.47	-6.87	17.84	-10.47	32.95
xly	63.85	-94.06	-77.36	-0.18	-71.26	-60.22	41.17	-13.17	48.94

Risk Neutral Data

- We next extracted data on the option prices for all out of the money options on all 9 underliers for 2009 July 21 with maturities between three months and a year.
- There were 167 option prices in all. We estimated the model *VGICA* with 27 parameters and the model *VGSSDICA* with 28 parameters to simultaneously determine the variance gamma laws of the 9 independent components with the exposure of each *ETF* to the factors as provided by the exposure matrix *A* displayed in Table 2.
- We thus have two complete specifications of the risk neutral law as linear mixtures of independent Lévy and Sato processes based on a variance gamma law at unit time.

Physical and Risk Neutral Correlations

- Our first exercise is to compare for the first time statistical and risk neutral correlations. For this purpose we generate three sets of 10000 readings on the 9 underliers from the estimated statistical and the two estimated risk neutral laws *VGICA* and *VGSSDICA*.
- We then compute the correlation matrix of the 9 underliers for each of the three simulations. We present in Table 3 the three correlation matrices.
- We observe as we expected that the risk neutral correlations are much higher than their statistical counterparts.

Table 3

Statistical Correlations between the 9 sector ETF's

	xlb	xle	xf	xli	xlk	xlp	xlu	xlv	xly
xlb	1.0000	0.7593	0.6582	0.8084	0.7390	0.6311	0.6224	0.6111	0.7270
xle	0.7593	1.0000	0.5341	0.6464	0.6246	0.5222	0.6639	0.5006	0.5609
xf	0.6582	0.5341	1.0000	0.7626	0.7326	0.6527	0.5258	0.6299	0.7983
xli	0.8084	0.6464	0.7626	1.0000	0.8164	0.7262	0.6062	0.6833	0.8496
xlk	0.7390	0.6246	0.7326	0.8164	1.0000	0.7260	0.6268	0.6646	0.8236
xlp	0.6311	0.5222	0.6527	0.7262	0.7260	1.0000	0.6584	0.7237	0.7428
xlu	0.6224	0.6639	0.5258	0.6062	0.6268	0.6584	1.0000	0.6164	0.5932
xlv	0.6111	0.5006	0.6299	0.6833	0.6646	0.7237	0.6164	1.0000	0.7066
xly	0.7270	0.5609	0.7983	0.8496	0.8236	0.7428	0.5932	0.7066	1.0000

Risk Neutral Correlations between the 9 sector ETF's as per VGICA

	xlb	xle	xf	xli	xlk	xlp	xlu	xlv	xly
xlb	1.0000	0.9827	0.9665	0.9769	0.9743	0.9816	0.9808	0.9574	0.9833
xle	0.9827	1.0000	0.9841	0.9845	0.9963	0.9809	0.9744	0.9773	0.9889
xf	0.9665	0.9841	1.0000	0.9721	0.9894	0.9732	0.9392	0.9493	0.9919
xli	0.9769	0.9845	0.9721	1.0000	0.9823	0.9806	0.9529	0.9507	0.9886
xlk	0.9743	0.9963	0.9894	0.9823	1.0000	0.9703	0.9558	0.9600	0.9876
xlp	0.9816	0.9809	0.9732	0.9806	0.9703	1.0000	0.9776	0.9738	0.9868
xlu	0.9808	0.9744	0.9392	0.9529	0.9558	0.9776	1.0000	0.9835	0.9585
xlv	0.9574	0.9773	0.9493	0.9507	0.9600	0.9738	0.9835	1.0000	0.9594
xly	0.9833	0.9889	0.9919	0.9886	0.9876	0.9868	0.9585	0.9594	1.0000

Risk Neutral Correlations between the 9 sector ETF's as per VGSSDICA

	xlb	xle	xf	xli	xlk	xlp	xlu	xlv	xly
xlb	1.0000	0.9673	0.9239	0.9489	0.9507	0.9706	0.9551	0.9111	0.9679
xle	0.9673	1.0000	0.9527	0.9516	0.9915	0.9651	0.9532	0.9537	0.9713
xf	0.9239	0.9527	1.0000	0.9088	0.9726	0.9297	0.8520	0.8634	0.9751
xli	0.9489	0.9516	0.9088	1.0000	0.9463	0.9597	0.8855	0.8679	0.9681
xlk	0.9507	0.9915	0.9726	0.9463	1.0000	0.9449	0.9112	0.9140	0.9734
xlp	0.9706	0.9651	0.9297	0.9597	0.9449	1.0000	0.9509	0.9314	0.9723
xlu	0.9551	0.9532	0.8520	0.8855	0.9112	0.9509	1.0000	0.9741	0.9007
xlv	0.9111	0.9537	0.8634	0.8679	0.9140	0.9314	0.9741	1.0000	0.8907
xly	0.9679	0.9713	0.9751	0.9681	0.9734	0.9723	0.9007	0.8907	1.0000

Construction of Hedge

- The target cash flow (tcf) is specified on defining the vector α and the matrix β of equation. This vector and matrix was randomly generated with β organized to have 5 positive and 4 negative eigenvalues.
- The vector α and the matrix β is displayed in Table 4.
- We employed the path space generated by *VGSSDICA* and constructed on this path space the cash flows to the hedging assets.
- The hedge cash flow is on a zero cost basis and for this we need to subtract the price.
- To ensure sample space risk neutrality the prices used were the sample means to the payoffs.

Table 4
Alpha Vector and Beta Matrix for Target Cash Flow

alpha	2.2225	0.1203	-0.3723	0.0988	-0.3669	-0.1686	1.5497	0.5105	-0.2561
	0.5670	-0.0718	0.0825	-0.0933	0.0858	-0.3342	-0.0987	-0.0036	-0.0824
	-0.0718	-0.0579	0.1781	0.2187	0.1270	-0.1655	-0.0141	-0.1316	0.2474
	0.0825	0.1781	0.4880	0.0628	0.0942	0.0603	0.0141	0.3479	-0.0574
	-0.0933	0.2187	0.0628	0.0205	-0.0628	-0.2002	0.0343	-0.0531	0.1941
Beta	0.0858	0.1270	0.0942	-0.0628	0.4560	0.1591	0.2599	-0.1412	0.1174
Matrix	-0.3342	-0.1655	0.0603	-0.2002	0.1591	0.4587	-0.1374	-0.0021	-0.2162
	-0.0987	-0.0141	0.0141	0.0343	0.2599	-0.1374	0.4846	0.1479	-0.1097
	-0.0036	-0.1316	0.3479	-0.0531	-0.1412	-0.0021	0.1479	0.0335	-0.0636
	-0.0824	0.2474	-0.0574	0.1941	0.1174	-0.2162	-0.1097	-0.0636	0.0495

- We also included the cash flow to the underlier and this gave us 22 hedging assets for each of 9 underliers.

Hedging Results

- We then first evaluated the ask price using *minmaxvar* at a stress of .75 with $\theta = 0$. The unhedged ask price *apuh* was

$$\begin{aligned} apuh &= 29.7975 \\ &= - \int_{-\infty}^{\infty} x d\Psi^{\gamma}(F_{(-tcf)}(x)). \end{aligned}$$

- The resulting unhedged cash flow *cfuh* is defined as

$$cfuh = apuh - tcf.$$

We then took positions in 21 *OTM* options on each of the nine underliers along with the underlier for three months. We constructed the residual cash flow (*rcf*) for the hedge θ as

$$rcf = \theta' H - tcf$$

where H is the sample matrix of hedge cash flows. The hedges are zero cost by construction and are

all self financed. We then constructed the ask price (aph) for the hedged cash flow

$$\begin{aligned} aph &= 7.9802 \\ &= - \int_{-\infty}^{\infty} x d\Psi^{\gamma}(F_{(\theta'H - tcf)})(x) \end{aligned}$$

- We then formed the hedged cash flow cfh by

$$cfh = aph + \theta'H - tcf.$$

- We observe that the hedge enables us to significantly reduce the ask price for the target cash flow using the same level of acceptability.
- We present in graphs of the histogram for the two unhedged and hedged cash flows over a sample space of 10000 paths on the 9 underliers modeled as linear mixtures of independent Sato processes as calibrated to the 9 option surfaces on 20090721.

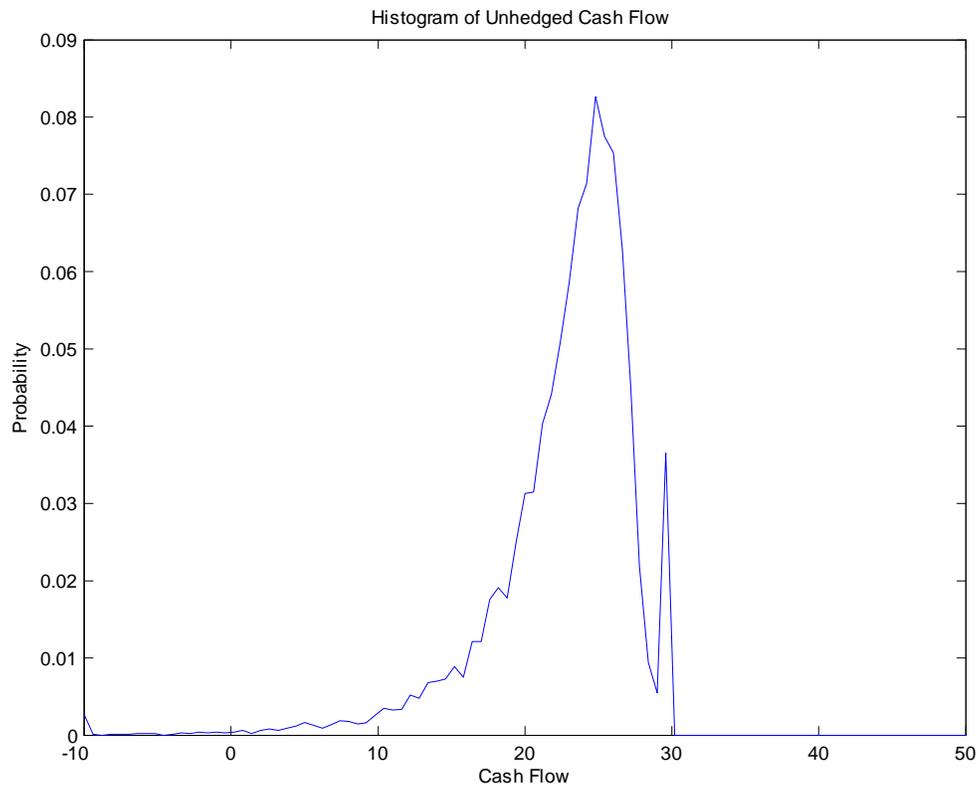


Figure 1: Histogram of unhedged cash flows with an ask price of 29.79.

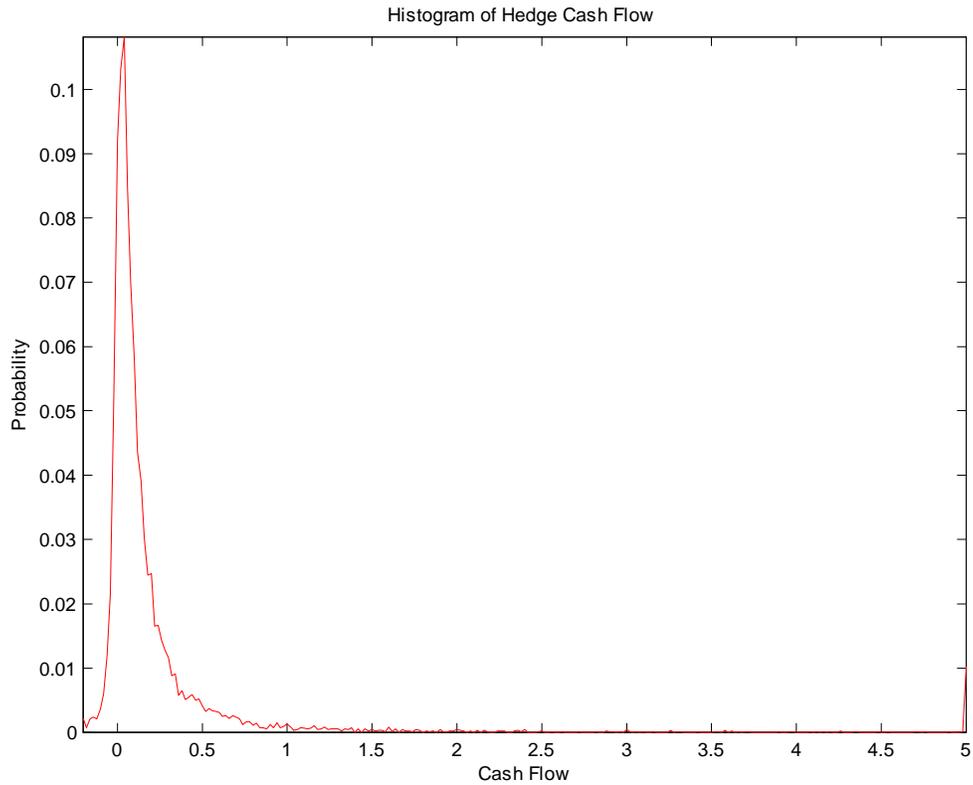


Figure 2: Histogram of hedged cash flow priced at 7.98 with a hedge position taken in 21 options for each of the 9 underliers along with the underlier itself. This is a set of 198 zero cost hedging assets.

Table 5

Cost of Hedge

Sector	Cost
xlb	0.5455
xle	0.2119
xlfi	0.9464
xli	0.2902
xlk	0.1477
xlp	-0.1026
xlu	0.5139
xlv	0.2402
xly	0.0375

- The cost of the hedge in total was 2.8307. The cost by sector is provided in Table 5.
- We present in Table 6 the precise hedge positions taken in all the options and underliers.

Table 6

Hedge Positions in 21 Out of the Money Options and the underlier for maturity 3 months

strike	P/C	xlb	xle	xlf	xli	xlk	xlp	xlu	xlv	xly
0.75	P	4.5925	1.8194	0.2963	2.5090	2.5031	-0.4794	9.9790	1.6879	2.3785
0.775	P	-1.6152	1.1514	-2.9564	-4.0867	-1.4656	-0.6647	-10.7183	-0.6860	-0.8081
0.8	P	2.2913	-4.1598	2.8671	2.8633	-1.0068	-1.4729	3.3972	1.5505	-1.8389
0.825	P	0.1688	3.1227	-0.2310	-2.3088	3.2153	0.5183	0.4924	-0.6767	1.4380
0.85	P	-0.9755	5.3059	0.9014	2.0639	-2.3413	0.5677	2.0982	1.0997	-0.9950
0.875	P	2.5994	-5.3263	1.2194	-0.9052	2.1684	-0.6073	0.6428	-0.5764	-1.2281
0.9	P	-0.6787	1.3170	1.1087	-0.4986	-0.5374	0.3990	3.1002	0.9880	1.2501
0.925	P	0.7934	0.0080	0.8821	1.4132	0.6110	-0.6663	-1.4748	-0.5459	-1.6192
0.95	P	0.8259	1.1369	1.2913	0.1193	-0.1537	0.0238	0.5783	1.2591	0.2845
0.975	P	1.1040	0.5326	1.8428	-0.1423	0.6950	-0.7029	1.3198	0.2870	-0.3820
1	C	0.5400	-2.2099	0.7585	0.8152	1.1407	0.6419	0.1112	-0.0800	-0.6713
1.025	C	1.5066	1.9724	2.6373	0.8546	-0.5331	-0.5506	0.2359	1.2803	0.4083
1.05	C	0.6959	-1.5130	2.0608	0.1855	2.2503	-0.4315	0.4747	-0.6091	-0.6723
1.075	C	1.3297	0.9522	3.5348	0.1772	-2.2083	0.7829	-1.7348	2.3734	-0.4831
1.1	C	2.6722	-2.7876	0.6956	1.3226	3.5492	-1.4083	3.1243	-2.2366	-0.8022
1.125	C	3.9857	4.8191	8.9503	-1.9009	-0.7380	1.1909	-1.6110	4.6155	7.8555
1.15	C	-5.4516	-8.8295	-1.3949	3.9766	1.8327	-1.7712	-3.1211	-2.8195	-11.2519
1.175	C	19.1746	20.3424	4.7143	-6.4690	1.7186	0.0603	20.0147	1.3232	38.5919
1.2	C	-15.2501	347.8769	6.7715	9.9710	19.3380	3.9919	-33.3232	1.9873	-72.2486
1.225	C	11.2210	437.6256	-1.5936	-14.0622	-17.1515	-3.2589	76.1461	-3.1698	44.0100
1.25	C	4.5303	431.3317	25.6387	46.8324	-15.8373	-3.5527	484.0120	4.0598	323.6196
0	C	12.5527	3.6580	3.9789	0.6762	2.4006	-3.9598	12.3758	2.6352	0.7210

Conclusion

- We illustrate the hedging of a complex position on multiple underliers using the options surface of all underliers in a context where exact replication is not possible and residual risk must be held.
- Following Cherny and Madan (2010) we determine the hedge position to minimize the ask price that makes the post hedge residual risk acceptable to a prespecified level.
- It is shown that this requires one to use the risk neutral law on the set of underlying risks.
- A joint risk neutral law is estimated from multiple option surfaces by specifying the joint law as a linear combination of independent components.

- The independent components are estimated by an application of independent components analysis on time series data for the underliers.
- Once the exposure matrix has been estimated, one may employ the joint characteristic for the logarithm of all the underlying returns to calibrate the risk neutral law to the option surfaces taken together.

- The risk neutral law is compared to the statistical law and we show that risk neutral correlations are significantly greater than their statistical counterparts.
- This has been suspected but never really estimated given the difficulty in accessing joint risk neutral laws as only options on assets taken marginally are traded.
- With the additional hypothesis of a linear factor model estimated from time series analysis we calibrate a joint risk neutral law.
- We observe that the availability of a hedge significantly reduces the ask price for a preset level of acceptability.
- We envisage that tracking the post hedge ask price of a position may serve as an early warning indicator of when a liability is getting too expensive to hold giving one access to market signals for reducing certain positions.